Complex Analysis

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Contents

Abstract

Unofficial material. Use with caution :D!

Chapter 1

Review of Complex Numbers

1.1 General

Complex numbers can be represented in the form

$$z = x + iy$$

where $x, y \in \mathbb{R}$ and $i^2 = -1$

$$x = \Re(z)$$
$$y = \Im(z)$$

- The Real numbers are a subset of the Imaginary numbers

$$\mathbb{R} = \{z \in \mathbb{C} \text{ s.t. } y = 0\}$$

- The Purely Imaginary numbers are the numbers $z = iy$, where $y \in \mathbb{R}$

Definition 1 (Addition of Complex Numbers)
If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

Definition 2 (Multiplication of Complex Numbers)
If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1\dot{z}_2 = (x_1 + iy_1)(x_2 + iy_2) \quad = x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2$$

Example 1
Suppose that $z = x + iy$. Then

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$$

Check:

$$
\begin{aligned}
z\frac{1}{z} &= (x+iy)\left(\frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}\right) \\
&= \frac{x^2}{x^2+y^2} - \frac{y\cdot(-y)}{x^2+y^2} + i\left(x\frac{-y}{x^2+y^2} + y\frac{x}{x^2+y^2}\right) \\
&= 1
\end{aligned}
$$

The Complex Field $\mathbb{C}$

- Commutativity: $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$
- Associativity: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2)z_3 = z_1(z_2 z_3)$
- Identity $z + 0 = z$ and $z.1 = z$
- Inverses $z + (-z) = 0$ and $z.\frac{1}{z} = 1$ for $z \neq 0$.
- Distributive Law: $(z_1 + z_2)w = z_1 w + z_2 w$

Definition 3 (Powers of i)

$$
i^n = \begin{cases} i \text{ if } n \equiv 1 (mod 4) \\ -1 \text{ if } n \equiv 2 (mod 4) \\ -i \text{ if } n \equiv 3 (mod 4) \\ 1 \text{ if } n \equiv 0 (mod 4) \end{cases}
$$

Definition 4 (Complex Conjugate)
If $z = x + iy$, then its complex conjugate is defined as $\overline{z} = x - iy$
Important!
We say that z is *purely imaginary* iff $z = -\overline{z}$
We say that z is *real* iff $z = \overline{z}$

Definition 5

$$
\Re(z) = \frac{z + \overline{z}}{2}
$$

$$
\Im(z) = \frac{z - \overline{z}}{2i}
$$

Definition 6 (Absolute value of a complex number)
The modulus, or absolute value of a complex number is given by

$$
|z| = \sqrt{x^2 + y^2}
$$

Proposition 1
.

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
2. $\overline{z_1 z_2} = \overline{z_1}.\overline{z_2}$
3. $z.\overline{z} = |z|^2$
4. $|z_1 z_2| = |z_1||z_2|$
5. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$
6. $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$ if $z \neq 0$

Proof. .
Part 3

$$z.\overline{z} = (x+iy)(x-iy) = x^2 + y^2 + i(yx - xy) = x^2 + y^2 + 0i = x^2 + y^2 = |z|^2$$

Part 4

$$|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)} = z_1 z_2 \overline{z_1 z_2} = z_1 \overline{z_1}.z_2 \overline{z_2} = |z_1|^2 |z_2|^2 \implies |z_1 z_2| = |z_1||z_2|$$

Part 6

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2} = \frac{\overline{z}}{|z|^2}$$

□

Example 2 (Proof that if z_0 is a root of a polynomial, then $\overline{z_0}$ is also a root)

Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial with $a_n, a_{n-1}, \ldots, a_0 \in \mathbb{R}$ Let z_0 be a root of the polynomial. Then $\overline{z_0}$ is also a root.

Proof. Since z_0 is root, we have that

$$0 = f(z_0) = a_n z_0^n + a_{n-1} z_0^{n-1} + \ldots + a_1 z_0 + a_0$$

Taking conjugates of both sides, we get that

$$\begin{aligned}
0 &= \overline{a_n z_0^n} + \overline{a_{n-1} z_0^{n-1}} + \ldots + \overline{a_1 z_0} + \overline{a_0} \\
&= \overline{a_n}\,\overline{z_0^n} + \overline{a_{n-1}}\,\overline{z_0^{n-1}} + \ldots + \overline{a_1 z_0} + \overline{a_0} \\
&= a_n (\overline{z_0})^n + a_{n-1} (\overline{z_0})^{n-1} + \ldots + a_1 \overline{z_0} + a_0 \text{ since all the } a_i \text{ are real} \\
&= f(\overline{z_0}) \implies \overline{z_0} \text{ is a root of } f(z)
\end{aligned}$$

□

Inequalities The following are important inequalities for working with complex numbers

1. The Triangle Inequality:

$$||z| - |w|| \leq |z + w| \leq |z| + |w|$$

2. $|z| \geq |\Re z|$

3. $|z| \geq |\Im z|$

Proof. of the Triangle Inequality

$$\begin{aligned}
\text{Consider } |z+w|^2 &= (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\
&= |z|^2 + z\overline{w} + w\overline{z} + |w|^2 = |z|^2 + z\overline{w} + \overline{z\overline{w}} + |w|^2 = |z|^2 + 2\Re(z\overline{w}) + |w|^2 \\
&\leq |z|^2 + 2|z\overline{w}| + |w|^2 = |z|^2 + 2|z||\overline{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 \\
&= (|z|^2 + |w|^2) \\
&\implies |z+w| < |z| + |w| \text{ by taking the square roots}
\end{aligned}$$

□

1.2 Polar Form of Complex Numbers

Since we can represent x and y in polars as $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have that

$$z = x + iy = r\cos(\theta) + ir\sin(\theta) = r(\cos\theta + i\sin\theta)$$

Here

$$r = |z| \text{ is called the modulus of the complex number}$$

$$\theta = \tan\frac{y}{x} \text{ is called the argument of the complex number, written as } \arg(z)$$

Note" θ is not uniquely defined. If we demand that $-\pi < \theta \leq \pi$, then we say that θ is the *principle argument* .

Proposition 2 (Multiplying Numbers in Polar Form)

$$\text{Take } z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then

$$\begin{aligned}
z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)\right) \\
&= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \text{ by the theorem of trig addition}
\end{aligned}$$

So

$$|z_1 z_2| = |z_1||z_2| \text{ and } \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \text{ up to multiples of } 2\pi$$

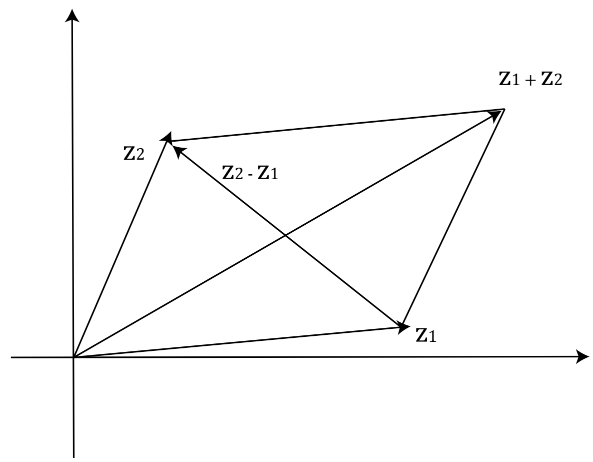
Corollary 1 (De Moivre's Theorem)

$$\text{If } z = r(\cos\theta + i\sin\theta) \text{ then } z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Geometric Interpretation of Complex Numbers Addition and Multiplication
.

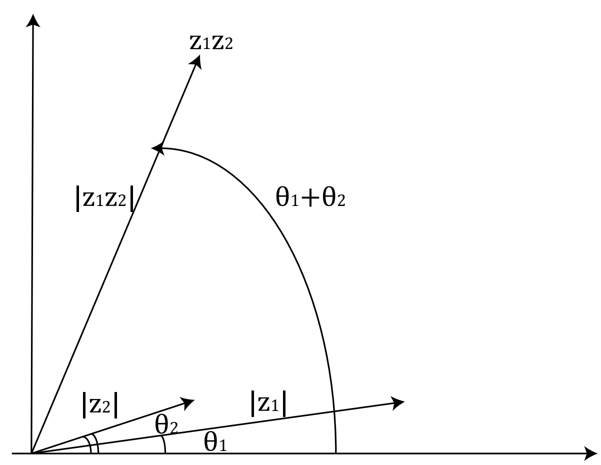
Addition

It is the same as vector addition - do it with the parallelogram and triangle laws.



Multiplication

Add the arguments and multiply the moduli.



1.3 Roots of Complex Numbers

Our aim is compute the n^{th} roots of equations of the form

$$z^n = a \text{ , where } z \in \mathbb{C} \text{ and } a \in \mathbb{C}$$

Write

$$z = \rho(\cos\phi + i\sin\phi) = \rho e^{i\phi}$$

Then

$$z^n = \rho^n(\cos(n\phi) + i\sin(n\phi)) = \rho^n e^{in\phi}$$

Write a as

$$a = |a|(cos\theta + i\sin\theta) = |a|e^{i\theta} \text{ where } \theta \text{ the is argument of } a$$

So now we have

$$\rho^n(\cos(n\phi) + i\sin(n\phi)) = |a|(\cos\theta + i\sin\theta) \text{ or } \rho^n e^{in\phi} = |a|e^{i\theta}$$

So comparing the moduli and arguments we get that

$$\rho^n = |a| \implies \rho = |a|^{\frac{1}{n}}$$

$$n\phi = \theta + 2m\pi \text{ for } m \in \mathbb{Z} \implies \phi = \frac{\theta + 2m\pi}{n}$$

Note that we need to restrict m so that we don't get infinitely many solutions. We take $0 \le m \le (n-1)$.

Example 3

Find the 4th roots of 1.

First of all, we write 1 as

$$1 = 1(\cos 0 + i\sin 0) = e^{i(0+2n\pi)}$$

And z^4 as

$$z^n = r^4(\cos(4\theta) + i\sin(4\theta)) = r^4 e^{i4\theta}$$

Since $r^4 = 1$, we conclude that $r = 1$. And for the arguments we have that

$$4\theta = 0 + 2n\pi \implies \theta = \frac{2n\pi}{4} \text{ for } n = 0, 1, 2, 3$$

Computing these explicitly gives

$$n = 0 \implies \theta_1 = 0 \implies z_1 = 1e^{i0} = 1$$
$$n = 1 \implies \theta_2 = \frac{2\pi}{4} = \frac{\pi}{2} \implies z_2 = 1e^{i\frac{\pi}{2}} = \left(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})\right) = (0 + 1i) = i$$
$$n = 2 \implies \theta_3 = \frac{4\pi}{4} = \pi \implies z_3 = 1e^{i\pi} = (\cos(\pi) + i\sin(\pi)) = (-1 + 0i) = -1$$
$$n = 3 \implies \theta_4 = \frac{6\pi}{4} = \frac{3\pi}{2} \implies z_4 = 1e^{i\frac{3\pi}{2}} = \left(\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2})\right) = (0 - 1i) = -i$$

Note that the roots all lie on the unit circle, and are equally spaced. In, general, the roots lie on a circle and are equally spaced. There are as many of them as the degree of z.

Chapter 2

Limits and Continuity

2.1 Sequences

Definition 7 (Convergence of a sequence of complex numbers)
The sequence $< z_n >$ of complex numbers converges to $w \in \mathbb{C}$ iff

$$\forall \varepsilon > 0, \exists N \text{ s.t. } n > N \implies |z_n - w| < \varepsilon$$

Definition 8 (Cauchy sequence)
The sequence $< z_n >$ is a Cauchy sequence if

$$\forall \varepsilon > 0, \exists N \text{ s.t. } n, m > N \implies |z_n - z_m| < \varepsilon$$

Theorem 1 (The combination theorem)

$$\text{If } \lim_{n \to \infty} z_n = w \text{ and } \lim_{n \to \infty} w_n = v \text{ then}$$

- $\lim_{n \to \infty} (z_n + w_n) = wv$
- $\lim_{n \to \infty} (z_n w_n) = wv$

The proofs are the same as in Analysis 1 and so we don't give them here.

Proposition 3
Let

$$z_n = x_n + iy_n \text{ where } x_n = \Re(z_n) \text{ and } y_n = \Im(z_n)$$

Then

$$\lim_{n \to \infty} z_n = w \text{ if and only if}$$

- $\lim_{n \to \infty} x_n = \Re(w)$
- $\lim_{n \to \infty} y_n = \Im(w)$

Proof. Part 1: $\implies$

Assuming that $\lim(z_n) = w$, we have that

$$\forall \varepsilon > 0, \exists N \text{ s.t. } n > N \implies |z_n - w| < \varepsilon$$

Consider

$$|\Re(z_n) - \Re(w)|$$

We have that

$$|\Re(z_n) - \Re(w)| = |\Re(z_n - w)| \leq |z_n - w| < \varepsilon \implies |\Re(z_n) - \Re(w)| < \varepsilon$$

So given $\varepsilon > 0$, choose the same N as for $\lim(z_n) = w$ and then we have that $\forall n > N$,

$$|\Re(z_n) - \Re(w)| < \varepsilon$$

which implies that

$$\lim_{n\to\infty} x_n = \Re(w)$$

We now repeat this proof for the second part. We consider this time

$$|\Im(z_n) - \Im(w)|$$

We have that

$$|\Im(z_n) - \Im(w)| \leq |z_n - w| < \varepsilon \implies |\Im(z_n) - \Im(w)| < \varepsilon$$

So given $\varepsilon > 0$, we choose the same N as in the definition we are assuming and we have that $\forall n > N$,

$$|\Im(z_n) - \Im(w)| < \varepsilon$$

which implies that

$$\lim_{n\to\infty} y_n = \Im(w)$$

Part 2: $\impliedby$

Assume that

$$\Re(z_n) \to \Re(w) \text{ as } n \to \infty$$
$$\Im(z_n) \to \Im(w) \text{ as } n \to \infty$$

So we have that

$$\forall \varepsilon > 0, \exists N_1 \text{ s.t. } n > N_1 \implies |\Re(z_n) - \Re(w)| < \frac{\varepsilon}{\sqrt{2}}$$

$$\forall \varepsilon > 0, \exists N_2 \text{ s.t. } n > N_2 \implies |\Im(z_n) - \Im(w)| < \frac{\varepsilon}{\sqrt{2}}$$

Now choose $N = \max(N_1, N_2)$, so that for $n > N$, both inequalities above hold.
Then $\forall \varepsilon > 0$, $n > N$, we have

$$\begin{aligned} |z_n - w| &= \sqrt{\Re(z_n - w)^2 + \Im(z_n + w)^2} \\ &< \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon \end{aligned}$$

which means that $\lim_{n\to\infty} z_n = w$ $\square$

Corollary 2

$$\text{If } \lim_{n\to\infty}(z_n) = w \text{ , then}$$

- $\lim_{n\to\infty} \overline{z_n} = \overline{w}$
- $\lim_{n\to\infty} |z_n| = |w|$

Note: $|z_n| = \sqrt{(\Re(z_n))^2 + (\Im(z_n))^2} \to \sqrt{(\Re(w))^2 + (\Im(w))^2} = |w|$

Proposition 4 (Cauchy $\iff$ convergent)
If z_n is convergent sequence, then it is Cauchy. The proof is the same as in Analysis 1. If z_n is Cauchy sequence, then it is convergent. And we actually need to prove this as it is not necessarily the same.

Cauchy $\implies$ convergent. Assume that $< z_n >$ is Cauchy, i.e. we have that

$$\forall \varepsilon > 0, \exists N \text{ s.t. } n, m > N \implies |z_n - z_m| < \varepsilon$$

Now, we have that

$$|z_n - z_m| \geq |\Re(z_n - z_m)| = |\Re(z_n) - \Re(z_m)|$$

and thus

$$|\Re(z_n) - \Re(z_m)| \leq |z_n - z_m| < \varepsilon \text{ i.e. } |\Re(z_n) - \Re(z_m)| < \varepsilon$$

which means that $\Re(z_n)$ is a Cauchy sequence of reals. By the General Principle of Convergence, it converges. Call its limit a, i.e.

$$\lim_{n\to\infty} \Re(z_n) = a$$

We repeat the argument for the imaginary part:

$$|z_n - z_m| \geq |\Im(z_n - z_m)| = |\Im(z_n) - \Im(z_m)|$$

and thus

$$|\Im(z_n) - \Im(z_m)| \leq |z_n - z_m| < \varepsilon \text{ i.e. } |\Im(z_n) - \Im(z_m)| < \varepsilon$$

which means that $\Re(z_n)$ is a Cauchy sequence of reals. By the General Principle of Convergence, it converges. Call its limit b, i.e.

$$\lim_{n\to\infty} \Im(z_n) = b$$

But now we can conclude that

$$\lim_{n\to\infty} = \lim_{n\to\infty} (\Re(z_n) + \Im(z_n)) = a + ib$$

i.e. the sequence is convergent. $\square$

2.2 (Topology) Open sets in the plane

Definition 9 (Open Disk)
The open disk centred at z_0 with radius $\varepsilon > 0$ is defined by

$$D(z_0, \varepsilon) = \{z \in \mathbb{C} \text{ s.t. } |z - z_0| < \varepsilon\}$$

So that's everything in the disk, including the centre, but without the boundary, i.e. without the circle.

Definition 10 (Punctured open disk)
The punctured (open) disk centred at z_0 with radius $\varepsilon > 0$ is defined by

$$D'(z_0, \varepsilon) = \{z \in \mathbb{C} \text{ s.t. } 0 < |z - z_0| < \varepsilon\}$$

So here we are excluding both the centre and the boundary of the disk.
Finally, we have

Definition 11 (Closed Disk)
The closed disk centred at z_0 with radius $\varepsilon > 0$ is defined by

$$\overline{D}(z_0, \epsilon) = \{z \in \mathbb{C} \text{ s.t. } |z - z_0| \le \varepsilon\}$$

So this set includes the disk with its centre and boundary.

Definition 12 (Open Set)
A set S is called open if for all $z \in S$, we can find an open disk $D(z, r)$ such that

$$D(z, r) \subseteq S$$

Remark: r depends on z.

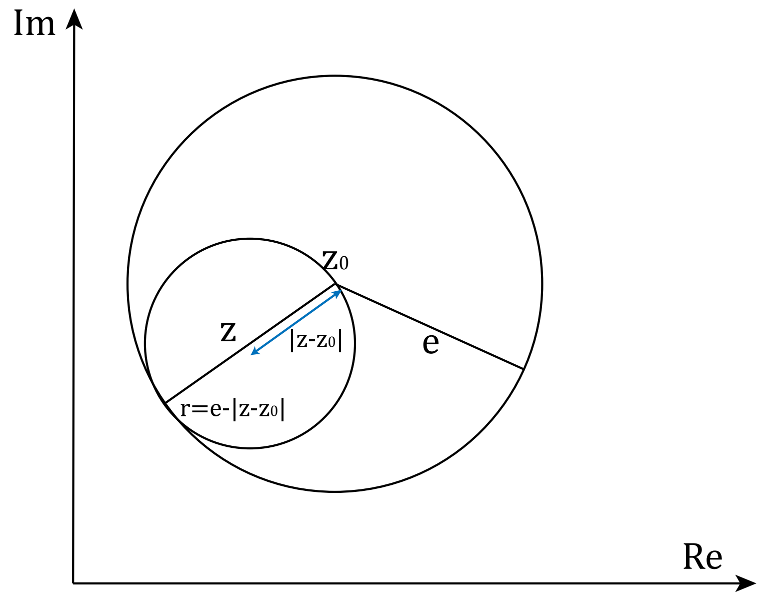
Method 1 (Showing that something is an open set)
.

1. We consider a point z is the set S which we want to prove to be open. We put an open disk D around this point, i.e. with this point at its centre, i.e. we take the disk to be $D(z, r)$, where r depends on z. Note that we generally need to find some expression for r since otherwise we won't be able to prove much.

2. With pick a point w in this disk and we write an expression for w, i.e. $|w - z| < r$, by our definition of an open disk.

3. We try to prove that this point also lies in the set S by looking at the properties of the elements of S and trying to deduce them from whatever expression we got from $|w - z| < r$

Example 4 (The open disk is an open set)
We consider the open disk $D(z_0, \varepsilon)$. We need to show that we can find another open disk, say $D(z, r)$ which is inside the original disk.
So our new disk is centred at some point z. We first need to write an expression for its radius, r.
Suppose that we picked the biggest disk centred at z which is included in the disk centred at z_0, so that the diameter of the disk centred at z is the same as the radius of the disk centred at z_0.



Then we have that

$$\varepsilon = r + |z - z_0| \implies r = \varepsilon - |z - z_0|$$

First all, the radius r is positive, which we can see as follows:
Since $z \in D(z_0, \varepsilon)$, we know that $|z - z_0| < \varepsilon$. But then $r = \varepsilon - |z - z_0| > 0$.
What we want to prove now is that

$$w \in D(z, r) \implies w \in D(z_0, \varepsilon)$$

where w is any point in the disk $D(z, r)$ which we constructed.
s we have

$$w \in D(z, r) \iff |w - z| < r = \varepsilon - |z - z_0|$$

and need to show

$$w \in D(z_0, \varepsilon) \iff |w - z_0| < \varepsilon$$

Start from the one we want to show

$$\begin{aligned}
|w - z_0| &= |w - z_0 + z - z| = |(w - z) + (z - z_0)| \\
&\leq |w - z| + |z - z_0| < r + |z - z_0| \\
&= \varepsilon - |z - z_0| + |z - z_0| = \varepsilon
\end{aligned}$$

So

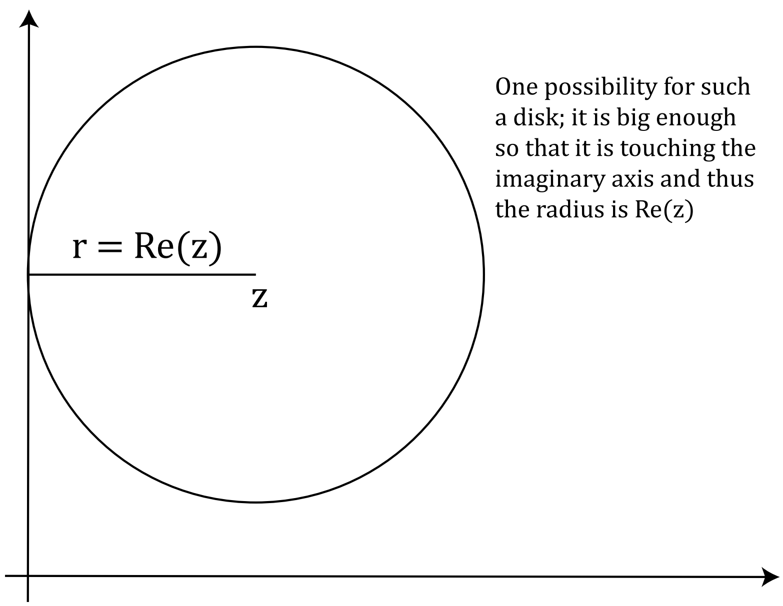
$$|w - z_0| < \varepsilon$$

which is what we wanted to show.

Example 5
Show that the set

$$A = \{z \in \mathbb{C}; \Re(z) > 0\}$$

is open
We start by picking any point $z \in A$ and drawing a disk around it. Let the radius of the disk be $\Re(z)$



We'll show that $D(z, \Re(z)) \subset A$
First of all, the disk has positive radius, since $z \in A \implies \Re(z) > 0$.
So now we take a point $w \in D$ and we are aiming to show that $w \in A$.
So we are assuming

$$w \in D(z, \Re(z)) \iff |w - z| < \Re(w)$$

and trying to prove that

$$w \in A \iff \Re(w) > 0$$

We have

$$
\begin{aligned}
&|\Re(w-z)| \le |w-z| < \Re(z) \\
&= |\Re(w)-\Re(z)| \le |w-z| < \Re(z) \\
&\implies |\Re(w)-\Re(z)| < \Re(z) \\
&\iff \Re(z)-\Re(w) \le |\Re(w)-\Re(z)| < \Re(z) \text{ by the triangle inequality} \\
&\implies \Re(z)-\Re(w) < \Re(z) \\
&\implies -\Re(w) < 0 \\
&\iff \Re(w) > 0
\end{aligned}
$$

Example 6 (A set which is not open)
The set

$$B = \{z \in \mathbb{C}; \Re(z) \ge 0\}$$

This set includes the imaginary axis. So suppose we took a point $z \in B$ which lies on the imaginary axis. Then any disk we draw around this point is not going to be fully contained in the set B, in fact exactly half if the disk will be outside B.
Generally, any similar set, with a non-strict inequality, is not open. We need to have a strict inequality. Then if we draw is disk around a point very close to the, say, imaginary axis, the disk is still contained in the set, even though its radius might be infinitesimal.

2.3 Functions

In the following definitions and theorems, we are assuming that

$$f : \Omega \to \mathbb{C}$$

where Ω is a subset of $\mathbb{C}$. We are also going to write are real part of the function $f(z)$ as $u(x,y)$ and the imaginary part as $v(x,y)$

Example 7

$$
\begin{aligned}
&f(z) = z^2 \iff f(x+iy) = (x+iy)^2 = x^2 + y^2 + 2ixy \\
&\text{so } f(x+iy) = u(x,y) + iv(x,y) \\
&\text{where } u(x,y) = \Re(f) = x^2 + y^2 \text{ and } v(x,y) = \Im(f) = 2xy
\end{aligned}
$$

Definition 13 (Limit of a function)

$$
\begin{aligned}
&\lim_{z \to z_0} f(z) = w_0 \iff \\
&\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |z - z_0| < \delta \text{ and } z \in \Omega \implies |f(z) - w_0| < \varepsilon
\end{aligned}
$$

Remark: $0 < |z-z_0| < \delta, z \in \Omega$ can be written as $z \in \Omega \wedge D'(z_0, \delta)$.

Definition 14 (Continuity at a point)
We say that f is continuous at z_0 if

$$z_0 \in \Omega \text{ and } \lim_{z \to z_0} f(z) = f(z_0)$$

Remark: The limit is unique.

Definition 15 (Continuity in an interval)
f is continuous on Ω if f is continuous at z_0 for all $z_0 \in \Omega$

Theorem 2

$$\lim_{z \to z_0} f(z) = w_0 \iff \left(\lim_{z \to z_0} \Re(f) = \Re(w_0)\right) \wedge \left(\lim_{z \to z_0} \Im(f) = \Im(w_0)\right)$$

Proof. Part 1: $\Longrightarrow$
We are assuming

$$\lim_{z \to z_0} f(z) = w_0 \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } z \in \Omega \wedge 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \varepsilon$$

To prove that

$$\lim_{z \to z_0} \Re(f) = \Re(w_0)$$

we consider

$$|\Re(f - w_0)| = |\Re(f) - \Re(w_0)| \leq |f(z) - w_0|$$

But then, given $\varepsilon > 0$, for the same $\delta > 0$ as in the definition we are assuming, we have that

$$0 < |z - z_0| < \delta \implies |\Re(f) - \Re(w_0)| < \varepsilon \iff \Re(f) \to \Re(w_0) \text{ as } z \to z_0$$

Similarly,

$$|\Im(f - w_0)| = |\Im(f) - \Im(w_0)| \leq |f(z) - w_0|$$

But then, given $\varepsilon > 0$, for the same $\delta > 0$ as in the definition we are assuming, we have that

$$0 < |z - z_0| < \delta \implies |\Im(f) - \Im(w_0)| < \varepsilon \iff \Im(f) \to \Im(w_0) \text{ as } z \to z_0$$

Part 2: $\Longleftarrow$
Assume:

$$\lim_{z \to z_0} \Re(f) = \Re(w_0)$$
$$\lim_{z \to z_0} \Im(f) = \Im(w_0)$$

So

$$\forall \varepsilon > 0, \exists \delta_1 > 0 \text{ s.t. } 0 < |z - z_0| < \delta_1 \implies |\Re(f) - \Re(w_0)| < \frac{\varepsilon}{2}$$
$$\forall \varepsilon > 0, \exists \delta_2 > 0 \text{ s.t. } 0 < |z - z_0| < \delta_2 \implies |\Im(f) - \Im(w_0)| < \frac{\varepsilon}{2}$$

Choose $\delta = \min\{\delta_1, \delta_2\} > 0$. Then, for $0 < |z - z_0| < \delta$, both inequalities above hold. So we have

$$0 < |z - z_0| < \delta \implies |\Re(f) - \Re(w_0)| < \frac{\varepsilon}{2} \text{ and } |\Im(f) - \Im(w_0)| < \frac{\varepsilon}{2}$$

So consider now

$$|f(z) - w_0| \leq |\Re(f(z)) - \Re(w_0)| + |\Im(f(z)) - \Im(w_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So we have shown that

$$\begin{aligned}&\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.}\\ &0 < |z - z_0| < \delta \implies |f(z) - w_0| < \varepsilon \text{ so } \lim_{z\to z_0} f(z) = w_0\end{aligned}$$

□

Corollary 3
f is continuous at z_0 iff

$$\begin{gathered}\Re(f) \text{ is continuous at } z_0\\ \Im(f) \text{ is continuous at } z_0\end{gathered}$$

Proposition 5 (The Combination Theorem)

$$\begin{aligned}\lim_{z\to z_0}(f(z) + g(z)) &= \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z)\\ \lim_{z\to z_0}(f(z).g(z)) &= (\lim_{z\to z_0} f(z)).(\lim_{z\to z_0} g(z))\\ \lim_{z\to z_0}\left(\frac{f(z)}{g(z)}\right) &= \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)} \text{ if } \lim(g(z)) \neq 0\end{aligned}$$

Proposition 6 (Sequential Definition of continuity of a function)
We say that f is continuous at z_0 iff for all sequences $< z_n >$ with $\lim_{n\to\infty} z_n = z_0$, we have that

$$\lim_{n\to\infty} f(z_n) = f(z_0)$$

The proof is the same as in Analysis 1, and is not examinable.

Proposition 7

$$\text{Let } p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \text{ be a polynomial}$$

Then it is continuous on the whole complex plane.

Proof. Let $z_0 \in \mathbb{C}$ Then

$$\begin{aligned}\lim_{z\to z_0}(p(z)) &= \lim_{z\to z_0}(a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0)\\ &= \lim_{z\to z_0}(a_n z^n) + \lim_{z\to z_0}(a_{n-1}z^{n-1}) + \ldots + \lim_{z\to z_0}(a_1 z) + \lim_{z\to z_0}(a_0) \text{ by The Algebra of limits}\\ &= a_n \lim_{z\to z_0} z^n + a_{n-1}\lim_{z\to z_0} z^{n-1} + \ldots + a_1 \lim_{z\to z_0} z + a_0 \text{ since the } a_i \text{ are constants}\\ &= a_n\left(\lim_{z\to z_0} z\right)^n + \ldots + a_1\left(\lim_{z\to z_0} z\right) + a_0 \text{ by the Combination Theorem}\\ &= a_n z_0^n + \ldots + a_1 z_0 + a_0\\ &= p(z_0)\end{aligned}$$

$\square$

Corollary 4
If $R(z)$ is a rational function, i.e.

$$R(z) = \frac{P(z)}{Q(z)} \text{ where } P, Q \in K[x]$$

then $R(z)$ is continuous whenever $Q(z) \neq 0$

Example 8

$$f(z) = Arg(z) \text{ for } z \neq 0 \text{ and } -\pi < Arg(z) \leq \pi$$

If $z_0 > 0$ for $z \in \mathbb{R}$, then $f(z)$ is continuous at z_0. But if $z_0 < 0$ for $z_0 \in \mathbb{R}$, then $f(z)$ is not continuous at z_0. Why is this so? We learned in Analysis 1 that, using the sequential definition, we can prove that a function is discontinuous at a point z_0 if we can find two sequences converging to this point, e.g.

$$\lim_{n\to\infty} z_n = z_0$$

$$\lim_{n\to\infty} w_n = z_0$$

But with

$$\lim f(z_n) \neq \lim f(w_n)$$

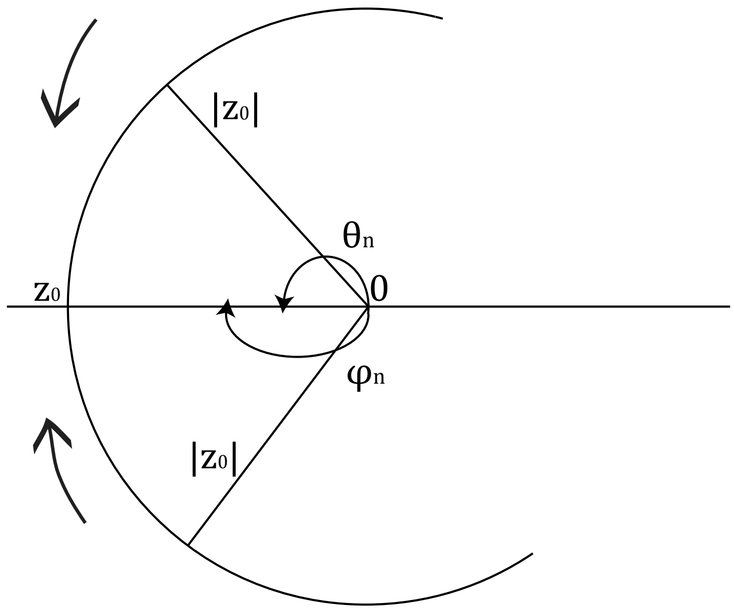
We are going to consider the sequences

$$z_n = |z_0|(\cos(\theta_n) + i\sin(\theta_n))$$

where θ_n is the sequences which tends to π, and which $\theta_n < \pi$

$$w_n = |z_0|(\cos(\psi_n) + i\sin(\psi_n))$$

where ψ_n is the sequence which tends to $-\pi$ and which $\psi_n > -\pi$



So we have that

$$z_n \to |z_0|(\cos(\pi) + i\sin(\pi)) = -|z_0| = -z_0$$

$$w_n \to |z_0|(\cos(-\pi) + i\sin(-\pi)) = -|z_0| = -z_0$$

But

$$f(z_n) = Arg(z_n) = \theta_n \to \pi$$

$$f(w_n) = Arg(w_n) = \psi_n \to -\pi$$

and $\pi \neq -\pi$. So the function is not continuous at z_0

Chapter 3

Differentiation

3.1 Derivatives and Important Results

Definition 16 (Derivative)
Let Ω be an open set in $\mathbb{C}$ and let $z_0 \in \Omega$. Let $f : \Omega \to \mathbb{C}$. If

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists, we call it the derivative of } f \text{ at the point } z_0$$

We write

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

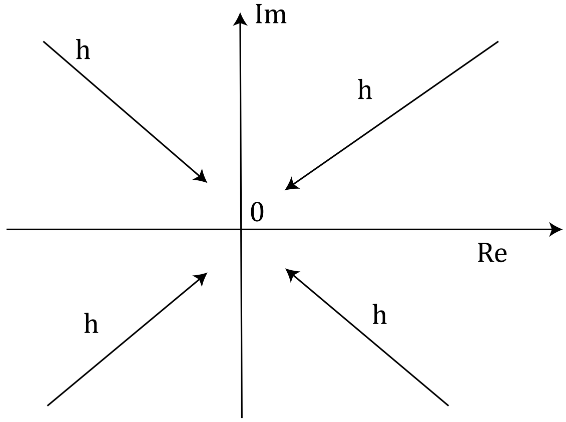
and we say that f is

- holomorphic
- analytic
- differentiable

at the point z_0
The alternative definition is

$$f'(z_0) \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

Note that h here is a complex number! So we can approach 0 from everywhere in the complex plane, including from the real or imaginary axis. In other words, for a complex function to be differentiable, it must be true that the limit exists and it unique no matter where we approach z_0 from.
Similarly, z_0 is complex number.
Remark: Since Ω is open and $z_0 \in \Omega$, $\exists \delta > 0$ s.t. $D(z_0, \delta) \subset \Omega$. Therefore, if $z \in D(z_0, \delta)$, then $f(z)$ makes sense. i.e. for $|h| < \delta$, $z = z_0 + h \in \Omega$



Terminology

Definition 17
If f is holomorphic for all $z_0 \in \Omega$, we say that f is holomorphic in Ω.

Definition 18
If S is any subset of $\mathbb{C}$, we say that f is holomorphic on S if we can find Ω open s.t. $S \subset \Omega$ s.t. f is holomorphic on Ω

Definition 19
If f is holomorphic on all $\mathbb{C}$, then we say that f is entire or integral function.

Example 9
Find the derivative of

$$f(z) = z^2$$

We have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} (z + z_0) = z_0 + z_0 = 2z_0$$

So $f'(z_0) = 2z_0 \implies f'(z) = 2z$

Example 10
Find the derivative of

$$f(z) = \overline{z}$$

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{\overline{z_0 + h} - \overline{z_0}}{h} \\ &= \lim_{h \to 0} \frac{\overline{z_0} + \overline{h} - \overline{z_0}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h} \end{aligned}$$

But we said that $h \in \mathbb{C}$, so this limit must exist and be unique to matter what h is. Let's pick two options for h and show that they won't give us the same answer.

If $h \in \mathbb{R}$. then we know that $\overline{h} = h$. So we have that

$$\lim_{h\to 0} \frac{\overline{h}}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

If h is purely imaginary, then we know that $\overline{h} = -h$, and so

$$\lim_{h\to 0} \frac{\overline{h}}{h} = \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0} -1 = -1$$

So we get different limits and hence the limit is not unique, which implies that the derivative doesn't exist. So the function is not differentiable.

Proposition 8
If f is holomorphic at z_0, then f is continuous at z_0.

Proof. Consider

$$f(z) = f(z_0) = \frac{f(z) - f(z_0)}{z - z_0}.(z - z_0)$$

Let $z \to z_0$. By definition

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \text{ since we are assuming that f is holomorphic}$$

And also,

$$\lim_{z\to z_0} (z - z_0) = 0$$

So, taking the limit of the LHS, we get that

$$\lim_{z\to z_0} f(z) - f(z_0) = 0 \implies f(z) \to f(z_0) \text{ as } z \to z_0$$

$\square$

Theorem 3 (The Combination Theorem)
Let f, g he holomorphic on Ω. Then

$$f + g \text{ is holomorphic on } \Omega \text{ and } (f+g)' = f' + g'$$
$$fg \text{ is holomorphic on } \Omega \text{ and } (fg)' = f'g + fg'$$
$$\text{If } g(z_0) \neq 0 \text{ , then } \frac{f}{g} \text{ is holomorphic at } z_0 \text{ and } \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Theorem 4 (The Chain Rule)
If $f : \Omega \to V$ and $g : V \to \mathbb{C}$ are holomorphic, then $g \circ f : \Omega \to \mathbb{C}$ is holomorphic and

$$(f \circ g)'(z) = f'(g(z))g'(z)$$

Remarks:

$$f(z) = c \implies f'(z) = 0$$

$$f(z) = z^n \implies f'(z) = nz^{n-1}$$

If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ then $p'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \ldots + a_1$

Example 11
Another function which is not holomorphic.

$$f : \mathbb{C} \to \mathbb{C}, f(z) = |z|^2$$

Consider

$$\begin{aligned}
\frac{f(z_0+h)-f(z_0)}{h} = \frac{|z_0+h|^2-|z_0|^2}{h} &= \frac{(z_0+h)\overline{(z_0+h)}-z_0\overline{z_0}}{h} \\
&= \frac{(z_0+h)(\overline{(z_0)}+\overline{(h)})}{h} = \frac{z_0\overline{z_0}+h\overline{z_0}+\overline{h}z_0+h\overline{h}-z_0\overline{z_0}}{h} \\
&= \frac{h\overline{z_0}+\overline{h}z_0+h\overline{h}}{h} = \overline{z_0}+\frac{\overline{h}}{h}z_0+\overline{h}
\end{aligned}$$

To find the derivative, we need to take the limit of this expression as $h \to 0$. However, once again, considering different h leads to different results and to the conclusion that f is not holomorphic since the limit is not unique.
For example, consider $h \in \mathbb{R}$. Then, since $\overline{h} = h$, we have that

$$\lim_{h\to 0}\left(\overline{z_0}+\frac{\overline{h}}{h}+\overline{h}\right) = \overline{z_0}+z_0+h$$

On the other hand, if h is purely imaginary, so that $\overline{h} = -h$, we have that

$$\lim_{h\to 0}\left(\overline{z_0}+\frac{\overline{h}}{h}+\overline{h}\right) = \overline{z_0}-z_0-h$$

Since the results are different, the function is not holomorphic!
Note that there is one point where the function is holomorphic, and that is at 0, since, if $z_0 = 0$, the limit is just 0, so it does exist.

Definition 20 (Partial Derivatives)
Recall the definition and notation for partial derivatives from Methods 2: If $u : \mathbb{R}^2 \to \mathbb{R}$, then

$$\frac{\partial u}{\partial x}(x_0, y_0) = \lim_{h\to 0}\frac{u(x_0+h, y_0)-u(x_0,y_0)}{h} = u_x(x_0,y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = \lim_{h\to 0}\frac{u(x_0, y_0+h)-u(x_0,y_0)}{h} = u_y(x_0,y_0)$$

3.2 The Cauchy-Riemann Equations

Theorem 5 (Cauchy-Riemann equations)

$$\text{Let } f(z) = u(x, y) + iv(x, y)$$

where $z = x + iy$ and $u(x, y) = \Re(f)$ and $v(x, y) = \Im(f)$.
Assume that f is holomorphic at z_0 and that u and v have partial derivatives at (x_0, y_0).
Then

$$\begin{aligned} f'(z_0) &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) \end{aligned}$$

Hence, u and v satisfy the following equations know as *The Cauchy-Riemann Equations*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

Remarks:

- The CRE show that if we have two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$, then it is very unlikely that $u + iv$ is holomorphic. So the CRE impose restrictions on u and v.
- The CRE are necessary but not sufficient conditions for $u + iv$ to be holomorphic.

Example 12
Let the function

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

So here

$$u(x, y) = x^2 - y^2 \text{ and } v(x, y) = 2xy$$

And they satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} = 2y$$

Proof. In the following proof we are going to use the fact that

$$\lim_{z \to z_0} (\Re(f(z))) = \Re\left(\lim_{z \to z_0} f(z)\right)$$

$$\lim_{z \to z_0} (\Im(f(z))) = \Im\left(\lim_{z \to z_0} f(z)\right)$$

We are going to consider two possibilities for h. We are going to consider h real and h purely imaginary and so arrive at the two forms of the derivative of f.

Part 1: $h \in \mathbb{R}$

Consider

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{u(x_0+h,y_0)+iv(x_0+h,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{h}$$

Consider only the real part of this expression

$$\Re\left(\frac{f(z_0+h)-f(z_0)}{h}\right) = \frac{u(x_0+h,y_0)-u(x_0,y_0)}{h}$$

We take te limit as $h \to 0$ of both sides and we have that

$$\begin{aligned}
\lim_{h\to 0}\Re\left(\frac{f(z_0+h)-f(z_0)}{h}\right) &= \lim_{h\to 0}\frac{u(x_0+h,y_0)-u(x_0,y_0)}{h} \\
\implies \Re\left(\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}\right) &= \frac{\partial u}{\partial x}(x_0,y_0) \text{ by defintion} \\
\implies \Re\left(f'(z_0)\right) &= \frac{\partial u}{\partial x}(x_0,y_0)
\end{aligned}$$

Similarly, from the imaginary part we get that

$$\Im\left(\frac{f(z_0+h)-f(z_0)}{h}\right) = \frac{v(x_0+h,y_0)-v(x_0,y_0)}{h}$$

We take te limit as $h \to 0$ of both sides and we have that

$$\begin{aligned}
\lim_{h\to 0}\Im\left(\frac{f(z_0+h)-f(z_0)}{h}\right) &= \lim_{h\to 0}\frac{v(x_0+h,y_0)-v(x_0,y_0)}{h} \\
\implies \Im\left(\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}\right) &= \frac{\partial v}{\partial x}(x_0,y_0) \text{ by defintion} \\
\implies \Im\left(f'(z_0)\right) &= \frac{\partial v}{\partial x}(x_0,y_0)
\end{aligned}$$

Combining the two results together, we get that

$$f'(z_0) = \Re f'(z_0) + \Im f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$$

Part 2: h is purely imaginary

Take $h = it$, since h is purely imaginary. Consider then

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{u(x_0,y_0+t)+iv(x_0,y_0+t)-u(x_0,y_0)-iv(x_0,y_0)}{it}$$

Again, we first take the real part of this expression. We have that

$$\Re\left(\frac{f(z_0+h)-f(z_0)}{h}\right) = \frac{v(x_0,y_0+t)-v(x_0,y_0)}{t}$$

Then again we take the limit as $h \to 0 \implies t \to 0$ and we get that

$$\begin{aligned}
&\lim_{h\to 0} \Re\left(\frac{f(z_0+h)-f(z_0)}{h}\right) = \lim_{t\to 0}\frac{v(x_0,y_0+t)-v(x_0,y_0)}{t}\\
&\implies \Re\left(\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}\right) = \frac{\partial v}{\partial y}(x,y_0)\\
&\implies \Re f'(z_0) = \frac{\partial v}{\partial y}(x,y_0)
\end{aligned}$$

And for the imaginary part, we have that

$$\Im\left(\frac{f(z_0+h)-f(z_0)}{h}\right) = \frac{-u(x_0,y_0+t)+u(x_0,y_0)}{t}$$

We take the limit as $h \to 0 \implies t \to 0$ and we get that

$$\begin{aligned}
&\lim_{h\to 0} \Im\left(\frac{f(z_0+h)-f(z_0)}{h}\right) = \lim_{t\to 0}\frac{-u(x_0,y_0+t)+u(x_0,y_0)}{t}\\
&\implies \Im\left(\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}\right) = -\lim_{t\to 0}\frac{u(x_0,y_0+t)-u(x_0,y_0)}{t}\\
&\implies \Im f'(z_0) = -\frac{\partial u}{\partial y}(x,y_0)
\end{aligned}$$

Combining these two, we get that

$$f'(z_0) = \Re f'(z_0) + \Im f'(z_0) = \frac{\partial v}{\partial y}(x_0,y_0) + i\frac{\partial u}{\partial y}(x_0,y_0)$$

Finally, the Cauchy-Riemann equations come from comparing the real and imaginary parts of the two results we have for $f'(z_0)$ $\square$

3.3 (Topology) Convex Sets, Paths and Polygonal Paths

Definition 21 (Line Segment)
The line segment between a and b is written as $[a,b]$ and is defined as

$$[a,b] = \{z = ta + (1-t)b, t \in [0,1]\}$$

Definition 22 (Convex Set)
A set S is called convex if $\forall a, b \in S$, the line segment $[a,b]$ is in S. i.e.

$$[a,b] \subseteq S$$

Example 13

The set $A = \{z : \Re(z) > 0\}$ is convex

Proof. As before, with open sets, we are going to proceed by picking points in the given set which form, in this case, the line segment and then picking a random point in the line segment and showing that it lies in the set. So we have:
Let $a, b \in A$. Then $\Re(a) > 0$ and $\Re(b) > 0$. We are going to show that $[a, b] \subseteq A$. That is, we pick any point $z \in [a, b]$ and we want to show that $z \in A$, i.e that $\Re(z) > 0$
Since $z \in [a, b]$, $\exists t \in [0, 1]$ s.t. $z = ta + (1 - t)b$, so we can write

$$\Re(z) = \Re(ta + (1-t)b) = \Re(ta) + \Re((1-t)b) = t\Re(a) + (1-t)\Re(b)$$

But since $t \in [0, 1]$, we have that $(1 - t) > 0$. Also, we are assuming that $\Re(a) > 0$ and $\Re(b) > 0$, and so the whole expression above is positive. That is, $\Re(z) > 0$ and so we can conclude that $z \in A$ □

Example 14

The disk $D(z_0, r)$ is convex

Proof. Let $a, b \in D(z_0, r)$ So we have that

$$|a - z_0| < r \text{ and } |b - z_0| < r$$

To show that $[a, b] \subseteq D(z_0, r)$, we pick a point $z \in [a, b]$ and we need to show that $z \in D(z_0, r)$, or that is to show that $|z - z_0| < r$. So,

$$z \in [a, b] \implies \exists t \in [0, 1] \text{ s.t. } z = ta + (1 - t)b$$

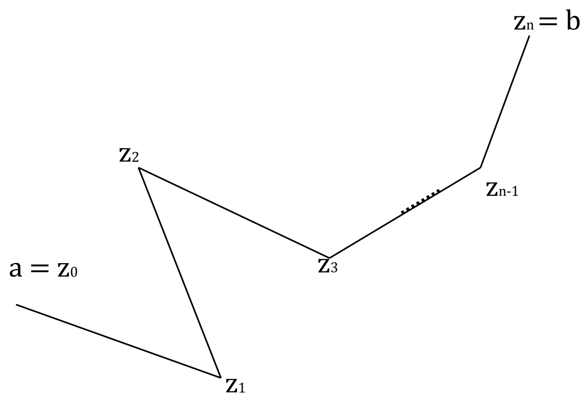
Now,

$$\begin{aligned}
|z - z_0| &= |ta + (1-t)b - z_0| = |ta + (1-t)b - (t + (1-t))z_0| \\
&= |ta - tz_0 + (1-t)b - (1-t)z_0| = |t(a - z_0) + (1-t)(b - z_0)| \\
&\leq |t(a - z_0)| + |(1-t)(b - z_0)| = t|a - z_0| + (1-t)|b - z_0| \\
&< t.r + (1-t)r = r
\end{aligned}$$

So $|z - z_0| < r$, which is what we wanted to prove. □

Definition 23 (Polygonal Path)
A polygonal path from a to b is a union of a finite sequence of line segments starting at a and finishing at b; i.e. a finite sequence of points $z_0, z_1, \ldots, z_n$ s,t, $z_0 = a$, $z_n = b$ and

$$[z_0, z_1] \cup [z_1, z_2] \cup \ldots \cup [z_{n-1}, z_n]$$



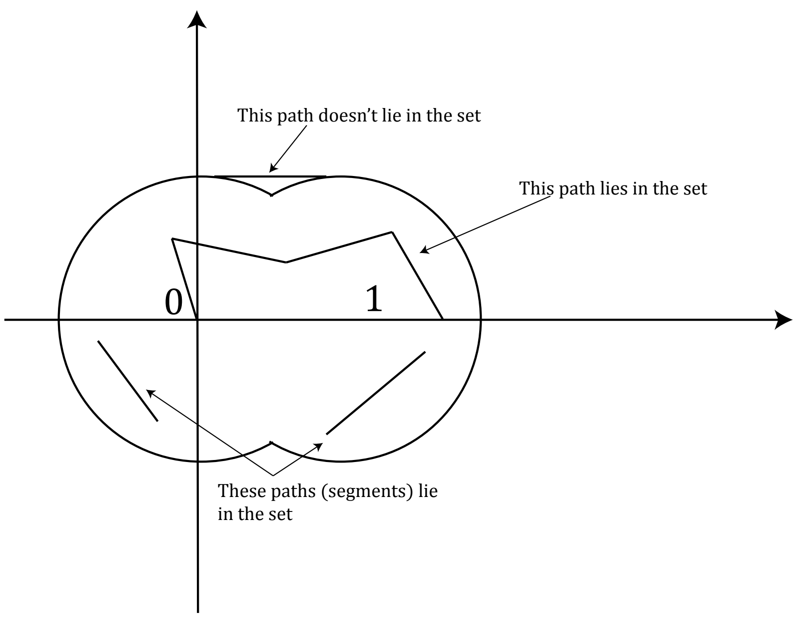
Definition 24 (Polygonally Connected Set)
A set S is called polygonally connected if $\forall a, b \in S$, we can find a polygonal path lying completely in S.

Example 15
Any convex set is polygonally connected

Example 16
Here is an example of a set which is not convex and not polygonally connected

$$D(0,1) \cup D(1,1)$$

Even though we can draw as many polygonal paths inside the set, we can also find ones which don't lie entirely in the set, while having their endpoints in the set.

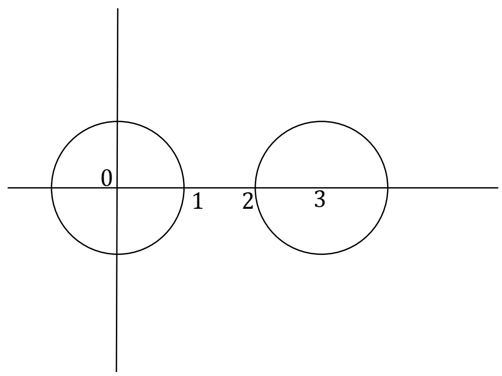


Definition 25 (Domain (Region))
A set Ω is called a domain or a region if it is open and polygonally connected.

Theorem 6
Let f be holomorphic in a domain Ω. Then

- If $f'(z) = 0$, $\forall z \in \Omega$, then $f = const$, i.e. $\exists k \in \mathbb{C}$ s.t. $f(z) = k$, $\forall z \in \Omega$
- If $\exists k \geq 0$ s.t. $|f(z)| = k$ $\forall z \in \Omega$, then $f = const$, i.e. $\exists c \in \mathbb{C}$ s.t. $f(z) = c$, $\forall z \in \Omega$.

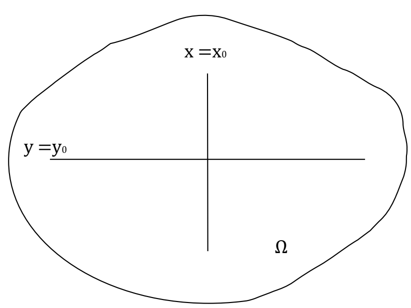
Remark: Ω needs to be polygonally connected. As an example of a set which is not polygonally connected and doesn't satisfy the conditions of the theorem, take $\Omega = D(0,1) \cup D(3,1)$. Let $f(z) = 1$ on the disk $D(0,1)$ and $f(z) = -1$ on the disk $D(3,1)$. Then $|f| = 1$ but the function is not constant.



Proof. *Part 1*
First of all, we are allowed to use the CRE, since f is holomorphic, so the assumption that $f'(z) = 0$ gives us that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Now consider the following diagram:



We have that

$$u(x, y_0) = const \text{ depending on } y_0, \text{ since } \frac{\partial u}{\partial x} = 0$$

$$v(x, y_0) = const \text{ depending on } y_0, \text{ since } \frac{\partial v}{\partial x} = 0$$
$$u(x_0, y) = const \text{ depending on } x_0, \text{ since } \frac{\partial u}{\partial y} = 0$$
$$v(x_0, y) = const \text{ depending on } x_0, \text{ since } \frac{\partial v}{\partial y} = 0$$

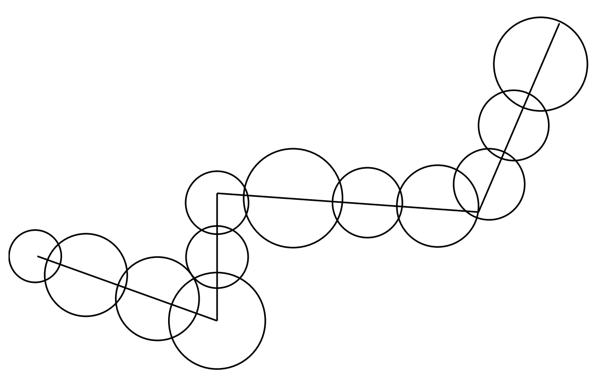
Now consider the following:
Let $z_0 \in \Omega$. Since Ω is open, we can find $r_0 > 0$ s.t. $D(z_0, r_0) \subseteq \Omega$. Take $z, w \in D(z_0, r_0)$. such that $[z_0, w]$ is a horizontal segment and $[w, z]$ is a vertical segment. Now, on the horizontal segment, the y-value is constant and on the vertical segment, the x-value is constant.
So we have that, for the horizontal line segment $[z_0, w]$:

$u(z_0, y) = u(w, y)$ and $v(z_0, y) = v(w, y)$ where y is fixed, so we can write $u(w) = u(z_0)$ and $v(w) = v(z_0)$

This implies that
$$f(w) = f(z_0)$$
And similarly, from the vertical line segment $[w, z]$, we have that:

$u(x, z) = u(x, w)$ and $v(x, z) = v(x, w)$ where x is fixed, so we can write $u(w) = u(z)$ and $v(w) = v(z)$

This implies that
$$f(w) = f(z)$$
So $(f(w) = f(z)) \wedge (f(w) = f(z_0)) \implies f(z) = f(z_0) = cosnt$. So we conclude that the function is constant of the disk $D(z_0, r)$.
But this only proves the result for the disk in our domain, and we want to prove that f is constant everywhere in the domain Ω.
So we consider the following: Since Ω is open, we can connect any point on it to any other point with a polygonal path. So we can get from any $a \in \Omega$ to any $b \in \Omega$ via a polygonal path. We are going to cover the whole path with overlapping disks and prove that since f is constant on all of the disks, it has to be the same constant everywhere and thus the same constant in the whole of Ω.



Let the disks be

$$D(z_0, r_0) \subseteq \Omega \text{ with } r_0 > 0$$
$$D(z_1, r_0) \subseteq \Omega \text{ with } r_1 > 0$$
$$\vdots$$
$$D(z_n, r_0) \subseteq \Omega \text{ with } r_n > 0$$

We construct a sequence of z_i on the polygonal path, with $r_i > 0$ s.t.

$$\bigcup_{i=1}^{n} D(z_i, r_i) \text{ convers the polygonal path}$$

Moreover, all the disks are in Ω, i.e. $\forall i, D(z_i, r_i) \subseteq \Omega$. Thus we know from the previous argument that $f = const$ on these disks, say

$$f(z) = c_i, \forall z \in D(z_i, r_i)$$

But since the disks overlap, i.e.

$$D(z_i, r_i) \cap D(z_{i+1}, r_{i+1}) \neq \varnothing$$

this is in fact the same constant; i.e. $c_i = c_{i+1}, \forall i$

Part 2

We are assuming that $\exists k \geq 0$ s.t. $|f(z)| = k$, $\forall z \in \Omega$. So we can write

$$|u(x,y) + iv(x,y)| = k \iff u^2 + v^2 = k^2$$

Differentiating w.r.t x, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \implies u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$$

Differentiating w.r.t y, we get

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \implies u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

Suppose that $k \neq 0$. We consider the system

$$uu_x + vv_x = 0$$

$$uu_y + vv_y = 0 \iff -uv_x + vu_x = 0 \text{ from the CRE}$$

This is a system in unknowns u_x, v_x and we can solve it by Cramer's Rule. We have

$$u_x = \frac{\begin{vmatrix} 0 & v \\ 0 & -u \end{vmatrix}}{\begin{vmatrix} u & v \\ v & -u \end{vmatrix}} = \frac{0}{-u^2 - v^2} = \frac{0}{-(v^2 + v^2)} = \frac{0}{-k^2} = 0 \text{ as } k \neq 0$$

$$v_x = \frac{\begin{vmatrix} u & 0 \\ v & 0 \end{vmatrix}}{\begin{vmatrix} u & v \\ v & -u \end{vmatrix}} = \frac{0}{-(v^2+u^2)} = \frac{0}{-k^2} = 0$$

So the conclusion is that $u_x = v_x = 0$ and thus

$$f'(z) = u_x + iv_x = 0 \implies f \text{ is const}$$

□

3.4 When if f holomorphic?

Theorem 7
Let $f(z) = u(x,y) + iv(x,y)$ be defined on an open set Ω containing $z_0 = x_0 + iy_0$. Assume the following:

- u and v are continuous on Ω
- $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous on Ω
- The CRE hold at z_0

Then, f is holomorphic at z_0

Reminder of the IVT:
If $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b) and continuous on $[a,b]$, then $\exists \xi \in (a,b)$ s.t. $f(a) - f(b) = f'(\xi)(b-a)$

Proof. Since Ω is open and $z_0 \in \Omega$, $\exists r > 0$ s.t. $D(z_0, r) \subseteq \Omega$
Take some $z \in D(z_0, r)$. Our aim will be to look at $\lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$
We have that

$$f(z)-f(z_0) = u(x,y)+iv(x,y)-u(x_0,y_0)-iv(x_0,y_0) = u(x,y)-u(x_0,y_0)+i[v(x,y)-v(x_0,y_0)]$$

where $u : D(z_0, r) \to \mathbb{R}$.
We polygonally connect z_0 to z, where z is some point in D. So we can write

$$\begin{aligned} f(z) - f(z_0) &= u(x,y) - u(x_0,y_0) + i[v(x,y) - v(x_0,y_0)] \\ &= u(x,y) - u(x,y_0) + u(x,y_0) - u(x_0,y_0) + i[v(x,y) - v(x,y_0) + v(x,y_0) + v(x,y_0)] \end{aligned}$$

We are going to apply the MVT 4 times to the 4 pairs of terms in the latter expression.
MVT on $u(x,y_0) - u(x_0,y_0)$. Remember that u is now a single-variable function since we are holding y_0 fixed, which is why we can apply the MVT. We have that

$$\exists \xi \in J_x \text{ where } J_x = [x_0, x] \text{ or } [x, x_0], \text{ s.t. } u(x,y_0) - u(x_0,y_0) = \frac{\partial u}{\partial x}(\xi, y_0)(x - x_0)$$

MVT on $u(x,y)-u(x,y_0)$. u is again single variable as we are holding x fixed. We have

$$\exists \eta \in J_y \text{ where } J_y=[y,y_0] \text{ or } [y_0,y],\ \text{s.t. } u(x,y)-u(x,y_0)=\frac{\partial u}{\partial y}(x,\eta)(y-y_0)$$

MVT on $v(x,y_0)-v(x_0,y_0)$; holding y_0 fixed.

$$\exists \xi' \in J_x \text{ where } J_x=[x_0,x] \text{ or } [x,x_0],\ \text{s.t. } v(x,y_0)-v(x_0,y_0)=\frac{\partial v}{\partial x}(\xi',y_0)(x-x_0)$$

MVT on $v(x,y)-v(x,y_0)$; holding x fixed.

$$\exists \eta' \in J_y \text{ where } J_y=[y,y_0] \text{ or } [y_0,y],\ \text{s.t. } v(x,y)-v(x,y_0)=\frac{\partial v}{\partial y}(x,\eta')(y-y_0)$$

We now take all of these results and plug them into the above expression of $f(z)-f(z_0)$. We get

$$f(z)-f(z_0)=\frac{\partial u}{\partial y}(x,\eta)(y-y_0)+\frac{\partial u}{\partial x}(\xi,y_0)(x-x_0)+i\left[\frac{\partial v}{\partial y}(x,\eta')(y-y_0)+\frac{\partial v}{\partial x}(\xi',y_0)(x-x_0)\right]$$

We rewrite this by adding and subtracting the same terms as:

$$\begin{aligned} f(z)-f(z_0)=&\left[\frac{\partial u}{\partial y}(x,\eta)-\frac{\partial u}{\partial y}(x_0,y_0)\right](y-y_0)+\frac{\partial u}{\partial y}(x_0,y_0)(y-y_0)\\ &+\left[\frac{\partial u}{\partial x}(\xi,y_0)-\frac{\partial u}{\partial x}(x_0,y_0)\right](x-x_0)+\frac{\partial u}{\partial x}(x_0,y_0)(x-x_0)\\ &+i\left(\left[\frac{\partial v}{\partial y}(x,\eta')-\frac{\partial v}{\partial y}(x_0,y_0)\right](y-y_0)+\frac{\partial v}{\partial y}(x_0,y_0)(y-y_0)\right)\\ &+i\left(\left[\frac{\partial v}{\partial x}(\xi',y_0)-\frac{\partial v}{\partial x}(x_0,y_0)\right](x-x_0)+\frac{\partial v}{\partial x}(x_0,y_0)(x-x_0)\right) \end{aligned}$$

Which we rearrange as

$$\begin{aligned} f(z)-f(z_0)=&\frac{\partial u}{\partial y}(x_0,y_0)(y-y_0)+\frac{\partial u}{\partial x}(x_0,y_0)(x-x_0)\\ &+i\left(\frac{\partial v}{\partial y}(x_0,y_0)(y-y_0)+\frac{\partial v}{\partial x}(x_0,y_0)(x-x_0)\right)\\ &+\left(\frac{\partial u}{\partial y}(x,\eta)-\frac{\partial u}{\partial y}(x_0,y_0)\right)(y-y_0)\ \text{A}\\ &+\left(\frac{\partial u}{\partial x}(\xi,y_0)-\frac{\partial u}{\partial x}(x_0,y_0)\right)(x-x_0)\ \text{B}\\ &+i\left(\frac{\partial v}{\partial y}(x,\eta')-\frac{\partial v}{\partial y}(x_0,y_0)\right)(y-y_0)\ \text{C}\\ &+i\left(\frac{\partial v}{\partial x}(\xi',y_0)-\frac{\partial v}{\partial x}(x_0,y_0)\right)(x-x_0)\ \text{D} \end{aligned}$$

And by the CRE, this is equal to

$$\begin{aligned}f(z)-f(z_0) &= -\frac{\partial v}{\partial x}(x_0,y_0)(y-y_0)+\frac{\partial u}{\partial x}(x_0,y_0)(x-x_0)+i\frac{\partial u}{\partial x}(x_0,y_0)(y-y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)(x-x_0)\\ &\quad +A+B+C+D\\ &= \frac{\partial u}{\partial x}(x_0,y_0)\left((x-x_0)+i(y-y_0)\right)+\frac{\partial v}{\partial x}(x_0,y_0)\left(i(x-x_0)-(y-y_0)\right)+A+B+C+D\\ &= \frac{\partial u}{\partial x}(x_0,y_0)(z-z_0)+i\frac{\partial v}{\partial x}(x_0,y_0)(z-z_0)+A+B+C+D\\ \Longrightarrow \frac{f(z)-f(z_0)}{z-z_0} &= \frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)+\frac{A+B+C+D}{z-z_0}\end{aligned}$$

We want to consider what happens as $z\to z_0$ and we expect to get that

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=u_x(x_0,y_0)+iv_x(x_0,y_0)$$

But to get this, we need to show that

$$\frac{A+B+C+D}{z-z_0}\to 0 \text{ as } z\to z_0$$

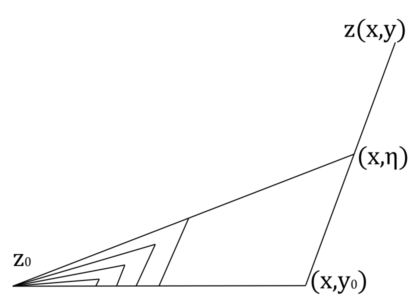
Consider just

$$\frac{A}{z-z_0}$$

We have that

$$\begin{aligned}\left|\frac{A}{z-z_0}\right| &= \frac{\left|\left(\frac{\partial u}{\partial y}(x,\eta)-\frac{\partial u}{\partial y}(x_0,y_0)\right)(y-y_0)\right|}{|z-z_0|}\\ &\le \frac{\left|\left(\frac{\partial u}{\partial y}(x,\eta)-\frac{\partial u}{\partial y}(x_0,y_0)\right)(z-z_0)\right|}{|z-z_0|}\\ &= \left|\frac{\partial u}{\partial y}(x,\eta)-\frac{\partial u}{\partial y}(x_0,y_0)\right|\end{aligned}$$

But this goes to 0 as $z\to z_0$ for the following reason:



We had that $\eta \in [y, y_0]$ or $[y_0, y]$ and as $z \to z_0$, we also have that $x \to x_0$ and $y \to y_0$, so as $z \to z_0$, $y \to y_0$ and so the length of the interval $J_y \to 0$. And since $\frac{\partial u}{\partial y}$ is continuous at (x_0, y_0) by assumption, we have that

$$\left|\frac{\partial u}{\partial y}(x, \eta) - \frac{\partial u}{\partial y}(x_0, y_0)\right| \to 0$$

Thus, the terms

$$\frac{A}{z - z_0} \text{ gives no contribution as we take the limit}$$

The proofs for the terms involving B, C and D are similar.
Thus,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = u_x(x_0, y_0) + iv(x_0, y_0) \implies f'(z_0) = u_x(x_0, y_0) + iv(x_0, y_0)$$

$\square$

Example 17
Is the following function holomorphic?

$$f(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} \text{ for } z \neq 0$$

We need to show that the partial derivatives for u and v exist and are continuous and satisfy the CRE. Here

$$u(x, y) = \frac{x}{x^2 + y^2} \text{ and } v(x, y) = -\frac{y}{x^2 + y^2}$$

So

$$u_x = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \text{ and } u_y = \frac{-2yx}{(x^2 + y^2)^2}$$

$$v_x = -\frac{-2xy}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2} \text{ and } v_y = -\frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = -\frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\text{So } u_x = v_y \text{ and } u_y = -v_x \implies \text{The CRE are satisfied}$$

All the partial derivatives are continuous as long as $(x, y) \neq 0$. We conclude by the Theorem, that f is holomorphic for $z \neq 0$
Note that

$$f(z) = \frac{x - iy}{x^2 + y^2} = \frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{z.\overline{z}} = \frac{1}{z}$$

We'll eventually prove that f'', f''' exist on Ω if f is holomorphic on Ω and this will imply that

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 v}{\partial x \partial y}$$

exist and are continuous

3.5 Harmonic and Conjugate Harmonic Functions. New Differential Operators

Definition 26 (Harmonic Functions)
We say that the function $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Example 18
If f is holomorphic, show that $u = \mathfrak{R} f$ and $v = \mathfrak{I} f$ are harmonic.
Since f is holomorphic, we can use the CRE. We have that

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = (u_x)_x = (v_y)_x = \frac{\partial^2 v}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial y^2} = u_{yy} = (u_y)_y = (-v_x)_y = -\frac{\partial^2 v}{\partial x \partial y}$$

As mixed partials are equal, we have that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \implies \text{u is holomorphic}$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} = v_{xx} = (v_x)_x = (-u_y)_x = -\frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial^2 v}{\partial y^2} = v_{yy} = (v_y)_y = (u_x)_y = \frac{\partial^2 u}{\partial x \partial y}$$

So again we get that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \implies \text{v is harmonic}$$

Definition 27 (Conjugate Harmonic)
If u and v are harmonic and satisfy the CRE, we say that they are conjugate harmonic

Example 19

$$\text{Let } u = 2x - x^3 + 3xy^2$$

Find its conjugate harmonic. Find f holomorphic s.t. $\mathfrak{R}(f) = u$. Write f as a function of z.
Part 1: Checking that u is harmonic.

$$u_x = 2 - 3x^2 + 3y^2 \implies u_{xx} = -6x$$

$$u_y = 6xy \implies u_{yy} = 6x$$

So $u_{xx} + u_{yy} = 0 \implies$ u is harmonic
Part 2: Finding the conjugate harmonic of u.
We are going to find v, the conjugate harmonic of u by using the CRE.

$$u_x = v_y = 2 - 3x^2 + 3y^2 \implies v(x,y) = \int (2 - 3x^2 + 3y^2)dy = 2y - 3x^2y + y^3 + c(x)$$

Differentiating this w.r.t. x, we get that

$$v_x = -6xy + \frac{d}{dx}c(x)$$

But we also have that

$$u_y = -v_x = 6xy \implies v_x = -6xy$$

$$\text{And thus } -6xy = -6xy + \frac{d}{dx}c(x) \implies \frac{d}{dx}c(x) = 0 \implies c(x) = k$$

$$\text{So } v(x,y) = 2y - 3x^2y + y^3 + k$$

Part 3: Finding f
We already showed in the previous example that for f holomorphic, the functions u and v s.t. $f = u + iv$ are conjugate harmonic. So the function we are looking for is

$$\begin{aligned} f(z) &= u + iv \\ &= (2x - x^3 + 3xy^2) + i(2y - 3x^2y + y^3 + k) = (2x + i2y) + (-x^3 + iy^3) + (3xy^2 + i(-3x^2y)) + ki \\ &= 2z - (x+iy)^3 + ki = 2z - z^3 + ki \end{aligned}$$

Definition 28

$$\frac{\partial f}{\partial x} = \frac{\partial(u+iv)}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$\frac{\partial f}{\partial y} = \frac{\partial(u+iv)}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

Definition 29

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

If f is holomorphic, then $\frac{\partial f}{\partial z}(z_0) = f'(z_0)$.

Example 20
Show that the CRE for $f = u + iv$ are equivalent to $\frac{\partial f}{\partial \bar{z}} = 0$

$$
\begin{aligned}
\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\right) \text{ from the CRE} \\
&= 0
\end{aligned}
$$

Chapter 4

Power Series

4.1 Review

Example 21 (Power Series)
A power series from Analysis 1:

$$\sum_{n=0}^{\infty} z^n$$

We consider the partial sums of this power series

$$\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z} \text{ for } z \neq 1$$

We have two possibilities:
If $|z| < 1$, then the series converges since $\lim_{n\to\infty} z^n = 0$. Then we say that the sum of the series is

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}$$

If $|z| > 1$, then $\lim_{n\to\infty} z^n = \infty$. Therefore, z^{N+1} is not convergent and so the series diverges.

Definition 30
Power series centred at 0 are of the form

$$\sum_{n=0}^{\infty} a_n z^n \text{ where } a_n \in \mathbb{C}, z \in \mathbb{C}$$

Definition 31
Power series centred at a are of the form

$$\sum_{n=0}^{\infty} a_n (z - a)^n \text{ where } a \in \mathbb{C}, z \in \mathbb{C}, a_n \in \mathbb{C}$$

Definition 32
If R is the radius of convergence, then $D(0, R)$ is the disk of convergence for power series centred at 0 and $D(a, R)$ for power series centred at a.

4.2 $\limsup$ and $\liminf$

Take a sequence $< b_n >$ with $b_n \geq 0$. Define

$$c_N = \sup\{b_N, b_{N+1}, b_{N+2}, \dots\} = \sup_{n \geq N} b_n$$

i.e. we are ignoring the first $N-1$ terms when finding the supremum.
If $< b_n >$ is bounded, then $c_N \in \mathbb{R}, c_N \geq 0$
If $< b_n >$ is unbounded, then $c_N = \infty$
We also have that

$$\{b_{N+1}, b_{N+2}, \dots\} \subseteq \{b_N, b_{N+1}, b_{N+2} \dots\} \implies \sup\{b_{N+1}, b_{N+2}, \dots\} \leq \sup\{b_N, b_{N+1}, b_{N+2} \dots\}$$

So $c_{N+1} \leq c_N$, i.e. $< c_N >$ is a decreasing sequence with numbers ≥ 0 or ∞ (bounded below by 0). Therefore, it is convergent, i.e. $\lim_{N\to\infty} c_N$ exists.
Remark: If $< b_n >$ is unbounded, then all $c_N = \infty$ and $\lim_{N\to\infty} c_N = \infty$

Definition 33

$$\limsup_{n\to\infty} b_n = \lim_{N\to\infty} c_N = \lim_{N\to\infty} \sup_{n \geq N} b_n$$

Example 22
Find the $\limsup$ of the sequence

$$b_n = 3 + \frac{(-1)^n}{n}$$

$b_1 = 3 - 1$
$b_2 = 3 + 1/2$
$b_3 = 3 - 1/3,$
$b_4 = 3 + 1/4$
$b_5 = 3 - 1/5$
$\sup_{n\geq 1} b_n = b_2$
$\sup_{n\geq 2} b_n = b_2 = 3 + 1/2$
$\sup_{n\geq 3} b_n = b_4$
$\vdots$
$\sup_{n\geq 2k} b_n = b_{2k} = 3 + \frac{1}{2k}$
$\sup_{n\geq 2k-1} b_n = b_{2k} = 3 + \frac{1}{2k}$
So we can write $< c_N >= \{3 + 1/2, 3 + 1/4, 3 + 1/4, 3 + 1/6, 3 + 1/6, \dots\}$. This is a decreasing sequence and

$$\lim_{N\to\infty} c_N = 3 \implies \limsup_{n\to\infty} b_n = 3$$

Remark: If $\lim_{n\to\infty} b_n = l$, then $\limsup_{n\to\infty} b_n = l$
Similarly, we can define

Definition 34

$$\liminf_{n\to\infty} b_n = \lim_{N\to\infty} \inf_{n\geq N} b_n$$

Remark: If $\lim_{n\to\infty} b_n = l$, then $\liminf_{n\to\infty} b_n = 1$

Lemma 1

$$\liminf \frac{b_{n+1}}{b_n} \leq \liminf \sqrt[n]{b_n} \leq \limsup \sqrt[n]{b_n} \leq \limsup \frac{b_{n+1}}{b_n}$$

Corollary 5

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = l \implies \lim_{n\to\infty} \sqrt[n]{b_n} = l$$

4.3 Important Theorems and Results about Complex Power Series

Definition 35 (Radius of Convergence with Hadamard's formula)
The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ or $\sum_{n=0}^{\infty} a_n z^n$ is given by Hadamard's Formula

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

Remark: If $\lim \sqrt[n]{|a_n|} = l$, then $R = \frac{1}{l}$.
This is *Cauchy's Root Test*, which says that

$$\text{If } \sum_{n=0}^{\infty} a_n z^n \text{ ,then we consider } \sqrt[n]{|a_n z^n|} = \sqrt[n]{|a_n|}|z| \to l|z| \text{ as } n \to \infty$$

And the root test says that

- If $l < 1$, then the series converges
- If $l > 1$, then the series diverges

Note that the sequence $< a_n >$ may not be a convergent sequence. This is when we we use Hadamard's Formula, as the sequence $< a_N >$ is not the same as $< a_n >$ and may be convergent. So if other tests are no good, use the lim sup and Hadamard. (Hadamard's formula and Cauchy's Root test involving lim sup are useful when we want to apply the root test but have a divergent sequence, i.e. one which might involve something like $(-1)^n$)

Theorem 8
There exists a unique $R \in [0, \infty]$ (i.e. we can define R to be 0 or ∞, too) s.t. for the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

we have

- if $|z| < R$, then the power series converges absolutely
- if $|z| > R$, then the power series diverges

Similarly, for the power series

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

- if $|z-a| < R$, then the series converges absolutely
- if $|z-a| > R$, then the series diverges

Remark: It is delicate to find out what happens on $|z| = R$ for centre at 0, or at $|z-a| = R$ for centre at a.

Proof. Define

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} \text{ and for simplicity assume that } R \neq 0, \infty$$

Let $c_N = \sup_{n \geq N} \sqrt[n]{|a_n|}$ (this is a decreasing sequence).
Let $l = \limsup \sqrt[n]{|a_n|} = \lim\limits_{N \to \infty} c_N$.
Part 1
We need to show that if $z \in \mathbb{C}$ with $|z| < R = \frac{1}{l} \iff l < \frac{1}{|z|}$, then $\sum\limits_{n=0}^{\infty} a_n z^n$ converges absolutely.
We pick $\varepsilon > 0$ s.t. $l + \varepsilon < \frac{1}{|z|} \implies (l+\varepsilon)|z| < 1$. Call $r = (l+\varepsilon)|z|$, so that $r < 1$.

$$\text{Since } l = \lim_{N \to \infty} c_N \text{ and } l + \varepsilon > l, \exists N \text{ s.t. } c_N < l + \varepsilon$$

So then we have that

$$c_N = \sup_{n \geq N} \sqrt[n]{|a_n|} < l + \varepsilon \implies \sqrt[n]{|a_n|} < l + \varepsilon, \forall n \geq N \implies |a_n| < (l+\varepsilon)^n$$

So now we consider the series

$$\sum_{n \geq N} |a_n z^n| = \sum_{n \geq N} |a_n||z|^n \leq \sum_{n \geq N} (l+\varepsilon)^n |z|^n = \sum_{n \geq N} r^n$$

But we showed that $r < 1$, so the series $\sum\limits_{n \geq N} r^n$ is the geometric series and it converges.
So by the Comparison Test, the series $\sum\limits_{n \geq N} |a_n z^n|$ converges, i.e. the series $\sum\limits_{n \geq N} a_n z^n$ converges absolutely.
And since adding a finite terms (the terms for $n < N$) doesn't affect convergence of the series, we conclude that

$$\sum_{n=0}^{\infty} a_n z^n \text{ converges absolutely}$$

Part 2

If $|z| > R$ then the series diverges.

We know for the n^{th} term test for convergence that $\sum\limits_{n=0}^{\infty} a_n z^n$ converges $\implies < a_n z^n > \to 0$ as $n \to \infty$. So we are going to prove the contrapositive of the statement above, i.e. we are going to show that the sequence $< a_n z^n > \not\to 0 \implies \sum\limits_{n=0}^{\infty} a_n z^n$ diverges. To do this, we will find an increasing sequence of natural numbers $a_{n_1} < a_{n_2} < \ldots$ with $n_1 < n_2 < \ldots$ such that

$$|a_{n_k} z^{n_k}| > 1$$

As before, we have that

$$|z| > R = \frac{1}{l} = \frac{1}{\limsup \sqrt[n]{|a_n|}} \implies l > \frac{1}{|z|}$$

We can find an $\varepsilon > 0$ s.t. $l - \varepsilon > \frac{1}{|z|} \implies 1 < (l - \varepsilon)|z|$

$$\text{Now, } c_1 = \sup_{n \geq 1} \sqrt[n]{|a_n|} > l - \varepsilon$$

so $l - \varepsilon$ is not an upper bound for $\{\sqrt[1]{|a_1|}, \sqrt[2]{|a_2|}, \sqrt[3]{|a_3|}, \ldots\}$.

$$\exists n_1 \text{ s.t. } \sqrt[n_1]{|a_{n_1}|} > l - \varepsilon$$

$$c_{n_1+1} = \sup_{n \geq n_1+1} \sqrt[n]{|a_n|} > l - \varepsilon$$

$$\exists n_2 \text{ s.t. } n_2 \geq n_1 + 1 \text{ with } \sqrt[n_2]{|a_{n_2}|} > l - \varepsilon$$

$$c_{n_2+1} = \sup_{n \geq n_2+1} \sqrt[n]{|a_n|} > l - \varepsilon$$

$$\exists n_3 \geq n_2 + 1 \text{ s.t. } \sqrt[n_3]{|a_{n_3}|} > l - \varepsilon$$

We continue in this way and deduce that

$$\sqrt[n_k]{|a_{n_k}|} > l - \varepsilon$$

So that

$$\sqrt[n_k]{|a_{n_k}|} > l - \varepsilon \implies |a_{n_k}| > (l - \varepsilon)^{n_k} \implies |a_{n_k} z^{n_k}| > \left[(l - \varepsilon)|z|\right]^{n_k} > 1$$

So by the n^{th} term test, the series diverges. $\square$

Example 23

A limit from Analysis last year: revision

$$\begin{aligned} \lim_{n \to \infty} \sqrt[n]{n} = \lim_{n \to \infty} n^{\frac{1}{n}} &= \lim_{n \to \infty} \left(e^{\log n}\right)^{\frac{1}{n}} \\ &= \lim_{n \to \infty} e^{\frac{\log n}{n}} = e^{\lim\limits_{n \to \infty} \left(\frac{\log n}{n}\right)} = e^0 = 1 \end{aligned}$$

Theorem 9
The power series

$$\sum_{n=0}^{\infty} a_n z^n \text{ for } |z| < R$$

defines a holomorphic function in the disk of convergence $D(0, R)$.
Moreover, its derivative is given by the power series

$$\sum_{n=1}^{\infty} n a_n z^{n-1}$$

obtained by differentiating the original series term by term.
Moreover, this series has the same radius of convergence as the original, R. So we have the holomorphic function and its derivative:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

Remark: The power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ are holomorphic inside $|z-a| < R$

Proof. The radius of convergence of

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ is given by } R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

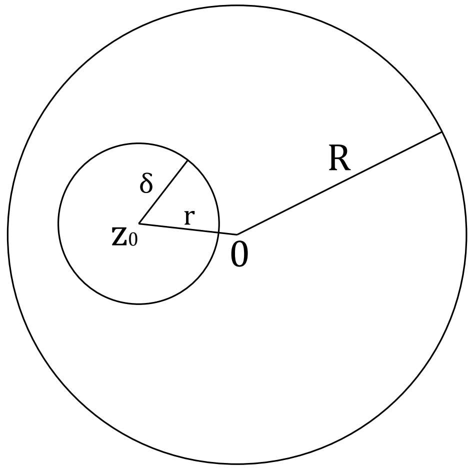
Then the radius of convergence of

$$\sum_{n=1}^{\infty} n a_n z^{n-1} \text{ is given by } R' = \frac{1}{\limsup \sqrt[n]{|n(a_n)|}} = \frac{1}{\limsup \sqrt[n]{n}\sqrt[n]{|a_n|}} = \frac{1}{\limsup \sqrt[n]{|a_n|}} = R$$

So these two series have the same radius of convergence.

$$\text{Now define } g(z) = \sum_{n=1}^{\infty} n a_n z^n \text{ , with } |z| < R$$

Fix $z_0 \in D(0, R)$, which is the disk of convergence of the power series.
We are aiming to show that $f'(z_0) = g(z_0)$ i.e, that the second power series is what we get by differentiating the first.



Let $z \in D(z_0, \delta)$ for $\delta << 1$, s.t. $D(z_0, \delta) \subseteq D(0, R)$.
Aim: Show that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) = 0$$

Label $|z_0| = r < R$.

$$\text{Define the partial sums } S_N(z) = \sum_{n=0}^{N} a_n z^n \left(\text{ s.t. } S_N \to \sum_{n=0}^{\infty} \text{ as } n \to \infty \right)$$

$$\text{Define the tail as } E_N(z) = \sum_{n>N} a_n z^n \text{ (s.t. } E_N \to 0 \text{ as } n \to \infty)$$

We also define the partial sums and the tail of $g(z_0)$ as

$$\sum_{n=0}^{N} n a_n z_0^{n-1} \text{ and } \sum_{n>N} n a_n z_0^{n-1} \text{ respectively}$$

And now we consider

$$\begin{aligned} \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) &= \frac{S_N(z) + E_N(z) - S_N(z_0) - E_N(z_0)}{z - z_0} - \sum_{n=0}^{N} n a_n z_0^{n-1} - \sum_{n>N} n a_n z_0^{n-1} \\ &= \left(\frac{S_N(z) - S_N z_0}{z - z_0} - \sum_{n=0}^{N} n a_n z_0^{n-1} \right) + \left(- \sum_{n>N} n a_n z_0^{n-1} \right) + \left(\frac{E_N(z) - E_N(z_0)}{z - z_0} \right) \end{aligned}$$

We are going to consider all the terms grouped as in the brackets.
Part 1 Consider first $-\sum_{n>N} n a_n z_0^{n-1}$.
Since $|z_0| < R$, the radius of convergence of $g(z)$, the series

$$\sum_{n=0}^{\infty} n a_n z_0^{n-1} \text{ converges, so } \exists N_1 \text{ s.t. } N > N_1 \implies \text{ the tails } \left| - \sum_{n>N} n a_n z_0^{n-1} \right| < \frac{\varepsilon}{3}$$

Part 2 Consider the terms $\frac{E_N(z)-E_N(z_0)}{z-z_0}$
We have that

$$\frac{E_N(z)-E_N(z_0)}{z-z_0} = \frac{\sum\limits_{n>N} a_n z^n - \sum\limits_{n>N} a_n z_0^n}{z-z_0} = \frac{\sum\limits_{n>N} a_n\left(z^n - z_0^n\right)}{z-z_0}$$
$$= \sum_{n>N} a_n \frac{z^n - z_0^n}{z-z_0} = \sum_{n>N} a_n\left(z^{n-1} + z^{n-2}z_0 + \ldots + z_0^{n-1}\right)$$

Now, we had that $z \in D(z_0, \delta)$ and this implies that $|z - z_0| < \delta$. We also defined that $|z_0| = r$. Combining these two and applying the triangle inequality, we get that

$$|z| < r + \delta$$

So if we consider the general form of each term appearing in the expansion above, we have that

$$|z^{n-i}z_0^{i-1}| = |z|^{n-i}|z_0|^{i-1} < (r+\delta)^{n-i}r^{i-1} < (r+\delta)^{n-i}(r+\delta)^{i-1} = (r+\delta)^{n-1}$$

Now plug everything back into the series (with modulus signs) to form an inequality:

$$\sum_{n>N} |a_n(z^{n-1}z_0 + z^{n-2}z_0^2 + \ldots + z_0^{n-1})| \le \sum_{n>N} n|a_n|(r+\delta)^{n-1}$$

The series

$\sum\limits_{n=0}^{\infty} n|a_n|(r+\delta)^{n-1}$ converges as $(r+\delta) < R$, which is the radius of convergence of $\sum\limits_{n=0}^{\infty} na_n z^{n-1}$

But we know that the tail of a convergent series $\to 0$, so we conclude that

$$\exists N_2 \text{ s.t. } N > N_2 \implies \sum_{n>N} |a_n(z^{n-1} + \ldots + z_0^{n-1})| < \frac{\varepsilon}{3}$$

Now take $\max\{N_1, N_2\}$ and fix $N > \max N_1, N_2$, The we have for this N (from Part 1 and Part 2) that

$$\left|\frac{E_N(z)-E_N(z_0)}{z-z_0}\right| < \frac{\varepsilon}{3} \text{ and } \left|\sum_{n>N} na_n z_0^{n-1}\right| < \frac{\varepsilon}{3}$$

Part 3 Consider the term $\frac{S_N(z)-S_N(z_0)}{z-z_0} - \sum\limits_{n=0}^{N} na_n z_0^{n-1}$
We have that

$$S_N(z) = \sum_{n=0}^{N} a_n z^n \text{ , which is a polynomial and its derivative is } S_N'(z) = \sum_{n=1}^{N} na_n z^{n-1}$$

$$\text{So } \lim_{z\to z_0} \frac{S_N(z)-S_N(z_0)}{z-z_0} = S_N'(z_0) = \sum_{n=1}^{N} na_n z_0^{n-1}$$

$$\text{So that } \exists \delta_1 > 0 \text{ s.t. } 0 < |z - z_0| < \delta_1 \implies \left|\frac{S_N(z)-S_N(z_0)}{z-z_0} - S_N'(z_0)\right| < \frac{\varepsilon}{3}$$

The last part of the proof is combining all together, we get that $\exists \delta_1 > 0$ s.t. $0 < |z - z_0| < \delta_1 \implies$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = g(z_0) = \sum_{n=0}^{\infty} n a_n z_0^{n-1}$$

$\square$

4.4 Examples - Finding the Radius of Convergence of Power Series

Example 24
Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} n^2 3^n z^n$$

We use the ratio test:

$$\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{(n+1)^2 3.3^n z.z^n}{n^2 3^n z^n} \right| = \left| \frac{3(n+1)z}{n^2} \right|$$
$$= \frac{3(n+1)}{n^2} |z| \to 3|z| \text{ as } n \to \infty$$

So if $3|z| < 1 \implies |z| < \frac{1}{3}$, we have that the series converges absolutely. If, on the other hand, $3|z| > 1 \implies |z| > \frac{1}{3}$, the series diverges. The radius of convergence is $R = \frac{1}{3}$.
What happens at $\frac{1}{3}$?
We plug in $z = \frac{1}{3}$ into $|n^2 3^n z^n|$ and we get $n^2 3^n \left(\frac{1}{3}\right)^n = n^2 \to \infty$ as $n \to \infty$.
But we know that if the series converges, then the sequence of the general terms tends to 0, while here at $z = \frac{1}{3}$ tends to ∞. So the series cannot converge for $z = \frac{1}{3}$. We conclude that it converges only inside the disk with radius $\frac{1}{3}$, but not on the circle.

Example 25
What is the radius of convergence of

$$\sum_{n=0}^{\infty} (3 + (-1)^n)^n z^n$$

We could try applying the root test, which gives us

$$\sqrt[n]{|(3 + (-1)^n)^n z^n|} = |(3 + (-1)^n) z| = (3 + (-1)^n)|z|$$

But this diverges as $n \to \infty$, since the sequence $(3 + (-1)^n)$ diverges. We can see this by choosing 2 subsequences, one of the odd terms and of the even terms. The fist one

converges to 4, while the second one converges to 2.
However, we can apply Hadamard's Formula, according to which

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

where $a_n = (3+(-1)^n)^n$, so $\sqrt[n]{|a_n|} = 3+(-1)^n$ and we showed that this is equal to 4 is n is even and to 2 if n is odd.
For lim sup, we define

$$c_N = \sup_{n \geq N}(3+(-1)^n) = 4$$

$$\limsup \sqrt[n]{|a_n|} = \lim_{N\to\infty} c_N = \lim(4) = 4$$

We conclude that

$$R = \frac{1}{4}$$

Example 26 (The exponential)
Find the radius of convergence of

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

We use the Ratio Test

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{\left|\frac{z^{n+1}}{(n+1)!}\right|}{\left|\frac{z^n}{n!}\right|} = \frac{|z^{n+1}|}{|z^n|}.\frac{n!}{(n+1)!} = |z|\frac{n!}{(n+1)n!} = \frac{|z|}{(n+1)} \to 0 \text{ as } n \to \infty$$

So the series converges $\forall z \in \mathbb{C}$. We conclude that $R = \infty$

Example 27 (Cosine)
Find the radius of convergence of

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

Again we use the ratio test

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{\frac{(-1)^{n+1}z^{2n+2}}{(2n+2)!}}{\frac{(-1)^n z^{2n}}{(2n)!}}\right| = |z|\frac{(2n)!}{(2n+2)!} = \frac{|z|}{(2n+2)(2n+1)} \to 0 \text{ as } n \to \infty$$

So again the radius of convergence is $R = \infty$

Example 28 (Sine)
Find the radius of convergence of

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Ratio Test:

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{\frac{(-1)^{n+1}z^{2n+3}}{(2n+3)!}}{\frac{(-1)^n z^{2n+1}}{(2n+1)!}}\right| = |z^2|\frac{(2n+1)!}{(2n+3)!} = \frac{|z|^2}{(2n+2)(2n+3)} \to 0 \text{ as } n \to \infty$$

So $R = \infty$

Example 29 (Exp)
From the power series definition of e^x, we have that

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = 1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \dots$$

But we also have that

$$\begin{aligned} e^{iz} = \cos(z) + i\sin(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right) \end{aligned}$$

We also have that

$$\begin{aligned} e^{-iz} = \cos(-z) + i\sin(-z) = \cos(z) - i\sin(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} - i\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) - i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right) \end{aligned}$$

This is how we get *Euler's Formulae*

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Another thing we can show about the exponential function is that its derivative is the same as itself. We have that

$$(e^z)' = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \text{ (by relabelling) } \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

Lemma 2
Let f be holomorphic in a region Ω with $f'(z_0) = f(z)$, $\forall z \in \Omega$.
Then $\exists k \in \mathbb{C}$ s.t. $f(z) = ke^z$, $\forall z \in \Omega$.

Proof. We are going to show that

$$\frac{f(z)}{e^z} = k$$

by differentiating it.

$$\left(\frac{f(z)}{e^z}\right)' = \frac{f'(z)e^z - (e^z)'f(z)}{(e^z)^2} = \frac{f(z)e^z - e^z f(z)}{e^{2z}} = 0$$

□

Example 30
Show that

$$e^{z+w} = e^z e^w, \forall z, w \in \mathbb{C}$$

Proof. Fix $w \in \mathbb{C}$ and define $f(z) = e^{z+w}$. Then $f'(z) = (z+w)'e^{z+w} = e^{z+w} = f(z)$ Using the lemma, $\exists k \in \mathbb{C}$ with $e^{z+w} = k.e^z$, $\forall z \in \mathbb{C}$. Plugging in $z = 0$, we get

$$e^w = ke^0 = k.1 = k$$

So $e^z + w = e^z.e^w$ $\square$

Useful to remember:

$$\begin{aligned} e^{\frac{\pi i}{2}} &= \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i \\ e^{i\pi} &= \cos(\pi) + i\sin(\pi) = -1 \implies e^{i\pi} + 1 = 0 \\ e^{\frac{i3\pi}{2}} &= -i \\ e^{z+2i\pi} &= e^z e^{2\pi i} = e^z(\cos(2\pi) + i\sin(2\pi)) = e^z \end{aligned}$$

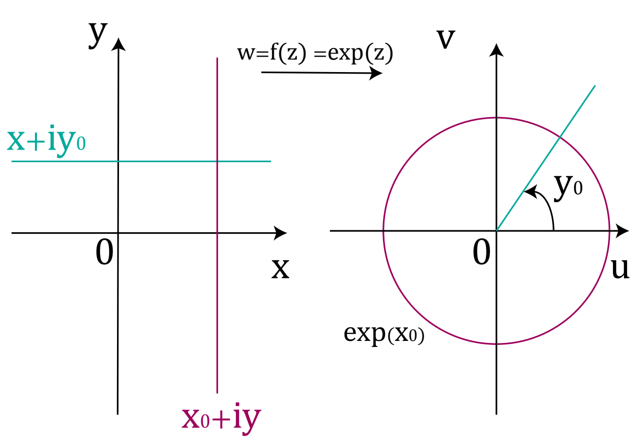
Chapter 5

Mapping Properties of Complex Functions. Conformal Maps

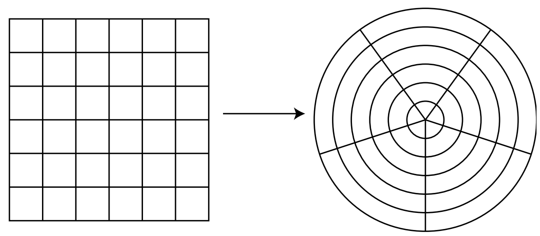
5.1 Mapping properties of e^z

For a complex function $f(z) \in \mathbb{C}$, $f : \mathbb{C} \to \mathbb{C}$ we get four dimensions which cannot be plotted easily. The way we plot complex functions is by introducing a z-plane and a w-plane. Then for z in the z-plane, we compute using $f(x_i y) = e^x(\cos y + i \sin y)$ the value of w which we represent on the w-plane. We plot the following:

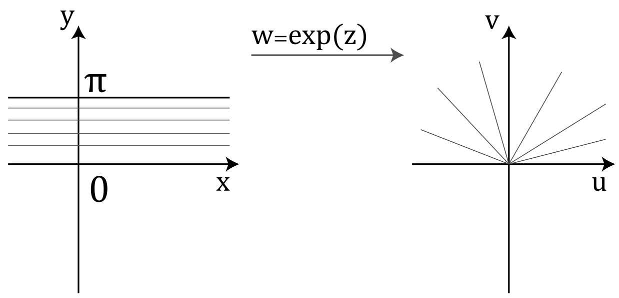
- The $z = 0$ value gives $e^z = 1$
- Horizontal lines of the type $z = x + iy_0$: We get $f(z) = e^x(\cos y_0 + i \sin y_0)$, which has a fixed argument y_0 and a variable length as x varies. Hence, horizontal lines are mapped onto rays emanating from the origin with angle y_0.
- Vertical lines of the type $z = x_0 + iy$: We get $f(z) = e^{x_0}(\cos y + i \sin y)$, which has a fixed modulus $|x_0|$ and a variable argument. Hence, vertical lines are mapped onto circles centred at the origin with radius e^{x_0}.



We notice that a grid system of horizontal and vertical lines maps to a series of circles and rays (centred and the origin and emanating from the origin). Thus the picture in the w-plane is a grid in polar coordinates. Thus orthogonality of the original vertical and horizontal lines is preserved (except at the origin).



Note that $f(z) = e^z$ is not injective in general because $e^{z+2i\pi} = e^z$. However, if we pick y_0, y_1 with $|y_0 - y_1| < 2\pi$, then it is injective. Consider the following proof:
Suppose that $e^{z_1} = e^{z_2}$ with $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then $|e^{z_1}| = e^{x_1}$ and $|e^{z_2}| = e^{x_2}$. Then $e^{z_1} = e^{x_1}(\cos y_1 + i \sin y_1) = e^{z_2} = e^{x_2}(\cos y_2 + i \sin y_2) \implies e^{x_1} = e^{x_2} \implies x_1 = x_2$ and $y_1 = y_2 + 2k\pi$. Hence, if $|y_2 - y_1| < 2\pi$ we have that $y_1 = y_2$ and thus the function is injective.
In fact, we have that e^z maps bijectively $\{z \in \mathbb{C} : 0 < \mathfrak{I}(z) < \pi\}$ in the z-plane to $\{w \in \mathbb{C} : \mathfrak{I}(w) > 0\}$, i.e. the horizontal strip of width 2π is mapped onto the upper half plane bijectively. This is the hyperbolic upper half plane.



5.2 Properties of $\log(z)$

Since e^z is not injective over the whole complex plane, its inverse may not exists. However, if we restrict domains, then it is possible that is does.
Given $w \in \mathbb{C}/\{0\}$, we find all solutions of $e^z = w$. To do this we let $w = |w|e^{i\theta}$. Then

$$\begin{aligned} e^z = w &\implies e^z = |w|(\cos(\theta) + i\sin(\theta)) \implies e^x(\cos y + i \sin y) = |w|(\cos(\theta) + i\sin(\theta)) \\ &\implies e^x = |w| \text{ and } y = \theta + 2k\pi \text{ for } k \in \mathbb{Z} \\ &\implies z = \log|w| + i\arg(w) \end{aligned}$$

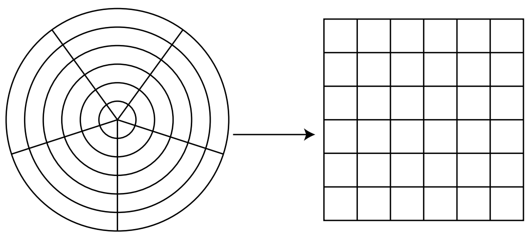
which is not unique in its imaginary part as it is determined up to $2k\pi$. This is why we define the principle logarithm.

Definition 36 (Principle Logarithm)
The principle branch of the complex logarithm is

$$Log(w) = \log|w| + iArg(w)$$

With this definition, the principle logarithm is the inverse of $\exp(z)$

Remark: The principle branch of the logarithm is not continuous on $\{z : z \in \mathbb{R}^- \cup 0\}$, where $Arg(w) = \pi$.
The function $Log(w)$ is holomorphic on $\mathbb{C}/\{z, z \leq 0\}$
Naturally, being the inverse function, log reverses the action of exp. So we get



5.3 Mapping Properties of $w = z^2$.

Here we have

$$z^2 = (x^2 - y^2) + 2ixy \implies u(x,y) = x^2 - y^2 \text{ and } v(x,y) = 2xy$$

5.3.1 Plotting Level Curves

To visualise this, we are going to use the technique of level curves.

- Fix $u = c$. We get that $u(x,y) = x^2 - y^2 = c$. Taking different values for c give us different rectangular hyperbolae. If $c = 0$, we get two lines from the origin. So rectangular hyperbolae in the z-plane are mapped onto lines $u = c$, i.e. vertical lines in the w-plane.

- Fix $v = k$. We get that $v(x,y) = xy = k$. Taking different values for k give us different hyperbolae. If $k = 0$, we get the origin. So hyperbolae in the z-plane are mapped onto lines $v = k$, i.e. horizontal lines in the w-plane.

We claim that in the z-plane our level curves from the sets $u = c$, $v = k$ intersect orthogonally. (i.e. their tangents at the points of the intersection are perpendicular). We can prove this:
Let $x^2 - y^2 = c$ and $2xy = k$. Suppose y is a function of x. Then differentiating w.r.t. x, we get

$$2xy = k \implies 2y + 2xy' = 0 \implies y' = -\frac{y}{x}$$
$$x^2 - y^2 = c \implies 2x - 2yy' = c \implies y' = \frac{y}{x}$$

Hence, if (x_0, y_0) is a point of intersection, the slopes are $\frac{x_0}{y_0}$ and $-\frac{y_0}{x_0}$. So the product of the slopes is -1 and thus the curves are orthogonal.

5.3.2 Plotting backwards

We can also reverse our process, i.e. we can look at what fixed shapes in the z-plane give in the w-plane.

- The circles $z = re^{i\theta}$. Then $f(z) = z^2 = (re^{i\theta})^2 = r^2\left[\cos(2\theta) + i\sin(2\theta)\right]$. So we get a circle again but this time its radius is r^2 rather than r and the circle is traced at twice the angular speed of the original one, i.e. going around the original circle once corresponds to going around the new circle twice.

- The rays with angle θ for a given value of θ. We get that $f(z) = f(\theta) = \theta^2$. So we get another ray in the same direction but with increased size.

- The shaded sector in the w-plane is subtended by an angle twice the size of the angle in the z-plane.

The general problem: Given two regions G and Ω, is there a holomorphic function $f : G \to \Omega$ which is bijective and $f^{-1} : G \to \Omega$ is holomorphic. If it exists, find it.

Definition 37 (Path/Curve)
A path or a curve in a region Ω is a continuous function $\gamma : [a, b] \to \mathbb{C}$ i.e. $\gamma(t) = x(t)+iy(t)$ where $x : [a, b] \to \mathbb{R}$ and $y : [a, b] \to \mathbb{R}$ are continuous functions and $x(t) + iy(t) \in \Omega$

If x' and y' exist $\forall t \in [a, b]$, then we say that γ is smooth.
We define $\gamma'(t) = x'(t) + iy'(t)$.
Take a curve parametrised by t and find a fixed point t_0. Then $\gamma'(t_0) = x'(t_0) + iy'(t_0)$. Then the vector $\gamma'(t_0)$ is parallel to he tangent line at γ at t_0, $\lim\limits_{t\to t_0} \frac{\gamma(t)-\gamma(t_0)}{t-t_0}$. So we can write the equation o f the tangent line. We know that $\arg(\gamma'(t_0))$ measures its inclination, i.e.

$$\text{slope of tangent line } = \frac{y'(t_0)}{x'(t_0)} = \tan(\arg(\gamma'(t_0)))$$

Now we take this curve and apply to it a mapping $w = f(z)$. Then the curve becomes $c(t) = f(\gamma(t))$ for $t \in [a, b]$
Assume that f is holomorphic and we can write it as $f = u + iv$. Then

$$c(t) = f(\gamma(t)) = u(\gamma(t)) + iv(\gamma(t)) = u(x(t), y(t)) + iv(x(t), y(t)) \text{ for } \gamma(t) = x(t) + iy(t)$$

$$\begin{aligned}\frac{dc}{dt} &= \frac{d}{dt}u(x(t),y(t)) + i\frac{d}{dt}v(x(t),y(t)) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + i\left(\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}\right)\\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{dy}{dt} = f'(z_0)\frac{dx}{dt} + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\frac{dy}{dt}\\ &= f'(z_0)\frac{dx}{dt} + if'(z_0)\frac{dy}{dt} = f'(z_0)(x'(t_0) + iy'(t_0))\\ &= f'(z_0)\gamma'(t_0)\end{aligned}$$

Then we can find that

$$c'(t_0) = \arg[f'(z_0)\gamma'(t_0)] = \arg(\gamma'(t_0))$$

5.4 Conformal Maps

Theorem 10
Given a complex function f s.t. f is holomorphic at z_0 and $f'(z_0) \neq 0$, then f preserves angles at z_0.
This means that if γ_1 and γ_2 are two smooth curves through z_0 and $f(\gamma_1)$, $f(\gamma_2)$ are the images through $f(z_0)$, then the angle between γ_1 and γ_2 is the same as the angle between $f(\gamma_1)$ and $f(\gamma_2)$.

Definition 38 (Conformal Map)
If $f : \Omega \to G$ preserves angles and

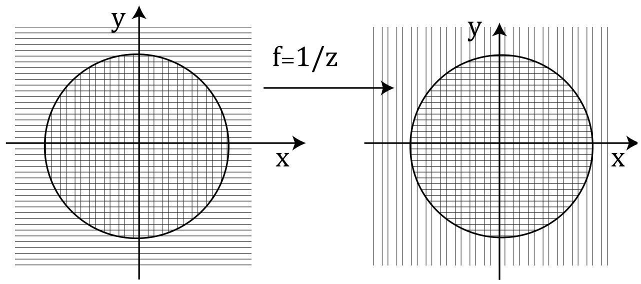
$$\lim_{z\to a}\frac{|f(z) - f(a)|}{|z - a|}$$

exists $\forall a \in \Omega$, then f is called a conformal map.
Remark 1: If f is holomorphic on Ω and $f'(z) \neq 0$ on Ω, then f is a conformal map. Note that this is the criterium we normally use to check if a map is conformal, rather than the official definition which involves taking the limit.
Remark 2: Domain restrictions can impact mappings. For example, consider again the mapping $f(z) = z^2$. We have that $f'(z) = 2z$ and so $f'(z) \neq 0$ for $z \neq 0$. Thus, f is conformal of $\mathbb{C}/\{0\}$ but not on the whole of $\mathbb{C}$.

Example 31 (Mapping properties of $f(z) = \frac{1}{z}$)
Consider the function $f(z) = \frac{1}{z}$. State the domain over which it is holomorphic and show that it maps circles and lines to circles and lines.
We have that $f : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$, $f(z) = \frac{1}{z}$ is bijective if we set $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.
We can write $f(z)$ as $f = |z^{-1}|e^{-i\theta}$.

- If $|z| = 1$, we get $f(z) = e^{-i\theta} \implies |f(z)| = 1$.
- If $|z| < 1$, we get $|f(z)| > 1$
- If $|z| > 1$, we get $|f(z)| < 1$

So everything inside the circle of radius 1 in the z-plane is mapped onto the exterior of the circle of radius 1 in the w-plane and vice versa. The circle itself is mapped onto the same circle.



Now, circles in the z-plane are given by the equation

$$a(x^2 + y^2) + bx + cy + d = 0 \text{ where } a, b, c, d \in \mathbb{R} \text{ for } a \neq 0$$

In fact all lines and circles are of this form (see discussion below).
We want to turn this equation into one in variables z and $\overline{z}$. We have:
$x^2 + y^2 = |z|^2 = z\overline{z}$. $x = \Re(z) = \frac{z+\overline{z}}{2}$ and $y = \Im(z) = \frac{z+\overline{z}}{2i}$. So we can write the above equation as

$$az\overline{z} + b\frac{z+\overline{z}}{2} + c\frac{z+\overline{z}}{2i} + d = 0$$

We now want to find the equation of this circle in the z-plane in the w-plane, i.e. after we have applied to it the function $w = f(z) = \frac{1}{z}$. So we write $w = \frac{1}{z} \implies z = \frac{1}{w}$ and then the equation becomes:

$$a\frac{1}{w\overline{w}} + \frac{b}{2}\left(\frac{1}{w} + \frac{1}{\overline{w}}\right) + \frac{c}{2i}\left(\frac{1}{w} - \frac{1}{\overline{w}}\right) + d = 0 \implies a + b\frac{w+\overline{w}}{2} + c\frac{\overline{w}-w}{2i} + dw\overline{w} = 0$$

Since $u = \frac{w+\overline{w}}{2}$ and $v = \frac{w-\overline{w}}{2i}$ in the w-plane, where $w = u + iv$ we get that the final equation is

$$a + bu - cv + d(u^2 + v^2) = 0$$

which is of the same form as the equation we had for the lines and circles in the z-plane. We conclude that $f = \frac{1}{z}$ maps lines and circles onto lines and circles.

The full classification is as follows:

$$a(x^2 + y^2) + bx + cy + d = 0$$

- Circle not passing through $(0,0) \implies a, d \neq 0$ is mapped to circle not passing through $(0,0)$
- Circle passing thought $(0,0) \implies a \neq 0$, $d = 0$ is mapped to line not passing through $(0,0)$

- Line not passing through $(0,0) \implies a = 0, d \neq 0$ is mapped to circle passing through $(0,0)$
- Line passing through $(0,0) \implies a, d = 0$ is mapped to line passing through $(0,0)$

Note how lines aren't necessarily mapped onto lines and circles aren't necessarily mapped onto circles.

Example 32
Let $w = f(z)$ $w = az + b$, where $a \neq 0$. Show that w is a bijection $f : \mathbb{C} \to \mathbb{C}$ and that is maps lines to lines and circles to circles.
Injective:

$$f(z_1) = f(z_2) \implies az_1 + b = az_2 + b \implies z_1 = z_2, \text{ for } a \neq 0$$

Surjective:

$$\text{Given } w \in \mathbb{C} \text{ we solve } w = az + b \implies z = \frac{w - b}{a} \text{ which is uniquely determined}$$

So the function is bijective.
To show that lines are mapped to lines and circles are mapped to circles, we write $a = re^{i\theta}$, so that $r = |a|$ and $\theta = \arg(a)$. Then $w = re^{i\theta}z + b$. Comparing z with $re^{i\theta}z + b$, we see that all points are only rotated and scaled and translated. Thus lines are mapped to lines and circles to circles.

5.5 Linear Fractional Transformations

We are going to analyse solution to

$$f(z) = w = \frac{az + b}{cz + d}$$

for $ad - bc \neq 0$ Such functions are called *linear fractional transformations*.
Case 1: $c = 0$, $d \neq 0$. Then

$$w = \frac{az + b}{d} = \left(\frac{a}{d}\right) z + \frac{b}{d}$$

This is in the same form as the linear function we considered in the example above, so we know that it maps lines to lines and circles to circles.
Case 2: $c \neq 0$.
We are going to first show that $w = f(z)$ is not a constant function.

$$w = \frac{az + b}{cz + d} = \frac{a}{c} \frac{z + \frac{b}{a}}{z + \frac{d}{c}} = \frac{a}{c} \frac{z + \frac{d}{c} - \frac{d}{c} + \frac{b}{a}}{z + \frac{d}{c}} = \frac{a}{c}\left(1 + \frac{\frac{b}{a} - \frac{d}{c}}{z + \frac{d}{c}}\right) = \frac{a}{c}\left(1 + \frac{1}{ac} \frac{-ad + dc}{z + \frac{d}{c}}\right)$$

Since $ad - bc \neq 0$, the fractional term appears and it is dependent on z, and hence the function is not constant.
Note that the function is not defined for $z = -\frac{d}{c}$.

We are now going to show that it is bijective and find the inverse function.
Using the relation

$$w = \frac{az+b}{cz+d}$$

we solve for z to get

$$w(cz+d) = az+b \implies cwz - az = b - dw \implies z = \frac{-dw+b}{cw-a}$$

so we get a unique solution for $w \neq \frac{a}{c}$. Hence, $w = f(z)$ is injective.
We define $f\left(-\frac{d}{c}\right) = \infty$ and $f(\infty) = \frac{a}{c}$. Then f is defined for all $\mathbb{C} \cup \{\infty\}$, i.e. $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is surjective.
Thus $w = f(z)$ is bijective and its inverse is given by

$$f^{-1}(w) = \frac{-dw+c}{cw-a} = \frac{dw-c}{-cw+a}$$

Example 33
Let f and g be linear fractional transformations given by

$$w = f(z) = \frac{az+b}{cz+d} \text{ and } g(w) = \frac{a'w+b'}{c'w+d'}$$

Where $ad - bc \neq 0$ and $a'd' - b'c' \neq 0$.
Compute $g \circ f$ and show that it is a linear fractional transformation.
We have that

$$(g \circ f)(z) = g(f(z)) = \frac{a'\frac{az+b}{cz+d} + b'}{c'\frac{az+b}{cz+d} + d'} = \frac{a'(az+b) + b'(cz+d)}{c'(az+b) + d'(cz+d)} = \frac{(a'a+b'c)z + (a'b+b'd)}{(c'a+d'c)z + (c'b+d'd)}$$

which is a linear fractional transformation for $\frac{c'a+d'c}{c'b+d'd} \neq 0$
Note: If we represent f by the coefficient matrix

$$F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and g by the coefficient matrix

$$G = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

Then $g \circ f$ is represented by GF.
Linear fractional transformations form a group. Also, $det(GF) = det(G)det(F) = (a'd' - b'c')(ad - bc) \neq 0$

Linear fractional transformations maps lines and circles to lines and circles. We can see this as follows: We write the linear fractional transformation in the form

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\left(1 + \frac{1}{ac}\frac{bc-ad}{z+\frac{d}{c}}\right)$$

Then

$$z \mapsto z + \frac{d}{c} = Z \text{ , which is a linear map}$$
$$Z \mapsto \frac{1}{Z} = W \text{ , whihc is a transfomation we have previously looked at}$$
$$W \mapsto \frac{a}{c} + \frac{1}{c^2}(bc - ad)W \text{ , which is a linear map}$$

Hence, linear fractional transformations are compositions of up to 3 known transformations which map lines and circles to circles and lines. Hence, they do the same.

Definition 39 (fixed point)
If f is a map and $f(z) = z$, then z is called a fixed point of f.

Example 34
Find all the fixed points of the linear fractional transformation $w = \frac{az+v}{cz+d}$, where $ad - bc \neq 0$.
We are looking for z s.t.

$$\frac{az+b}{cz+d} = z$$

Case 1: $c = 0$

$$\frac{a}{d}z + \frac{b}{d} = z \implies \left(\frac{a}{d} - 1\right)z + \frac{b}{d} = 0$$

- If $\frac{a}{d} \neq 1$, then we have a unique solution $z = -\frac{b/d}{a/d-1}$
- If $\frac{a}{d} = 1$ and $b = 0$, then $0 + \frac{b}{z} = z$. This is always true, i.e. $\forall z \in \mathbb{C}$, z is a fixed point and we have $w = \frac{az+0}{0z+d} = \frac{a}{d}z = z \implies$ this is the identity transformation.
- If $\frac{a}{d} = 1$ and $b \neq 0$, then there is no solution, i.e. no fixed points.

Case 2: $c \neq 0$.

$$\frac{az+b}{cz+d} = z \implies az + b = z(cz+d) = cz^2 + dz \implies 0 = cz^2 + (d-a)z - b$$
$$\implies z = \frac{-(d-a) \pm \sqrt{(d-a)^2 + 4bc}}{2c}$$

So w has up to 2 fixed points.
The conclusion is that if f is a linear fractional transformation and $f \neq Id$, it has at most 2 fixed points.

Corollary 6
If f is a linear fractional transformation and $f(z_1) = z_1$, $f(z_2) = z_2$, $f(z_3) = z_3$, with z_1, z_2, z_3 distinct, then $f(z) = z$, $\forall z \in \mathbb{C}$.

Example 35
Given $z_1, z_2, z_3 \in \mathbb{C}$ distinct and $w_1, w_2, w_3 \in \mathbb{C}$ distinct, find all linear fractional transformations T with $T(z_i) = w_i$

Consider the following transformation T. We claim that it is the one we are looking for.

$$\frac{\frac{w-w_1}{w-w_2}}{\frac{w_1-w_2}{w_1-w_3}} = \frac{\frac{z-z_2}{z-z_3}}{\frac{z_1-z_2}{z_1-z_3}} \implies \frac{w-w_1}{w-w_2}.\frac{z_1-z_2}{z_1-z_3} = \frac{z-z_2}{z-z_3}.\frac{w_1-w_2}{w_1-w_3}$$
$$\implies (w-w_2)(z_1-z_2)(z-z_3)(w_1-w_3) = (z-z_2)(w_1-w_2)(w-w_3)(z_1-z_3)$$

- Plugging in $z = z_3$ into the LHS, we get 0, so the RHS must also be 0. But since z_i are distinct, it must be true that if $z_3 = 0$, then $z_2 \neq 0$ and $z_1 \neq 0$ and since $w_1 \neq w_2$, the only term in the RHS that can be 0 is $(w - w_3)$. So we conclude that $w = w_3$, i.e. $z = z_3 \implies w = w_3$.

- Plugging in $z = z_2$ into the LHS, we get 0, so the RHS must also be 0. Since $w_1 \neq w_3$ and the z_i are also distinct, the only term that we can get to be 0 from the RHS is $(w - w_2)$. So we conclude that $z = z_2 \implies w = w_2$

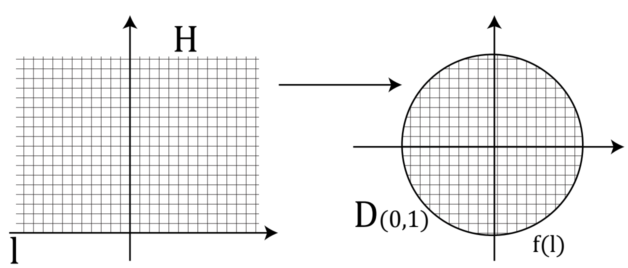
- Plugging in $z = z_1$ in the LHS, we get 0. We get

  $$(w-w_2)(z_1-z_2)(z_1-z_3)(w_1-w_3) = (z_1-z_2)(w_1-w_2)(w-w_3)(z_1-z_3) = 0 \implies (w-w_2)(w_1-w_3) = (w_1-$$

  This is linear, so it has only one solution. This is true for $w = w_1$, which is the unique solution. So once again we get that $z = z_1 \implies w = w_1$

Note: T is unique. Suppose that it wasn't and that we could find another transformation $S(z_i) = w_i$.
Consider $S \circ T^{-1}(w_i) = S(T^{-1}(w_i)) = S(w_i) = w_i$. So $S \circ T^{-1}$ has three fixed points and it is a linear fractional transformation $\implies S \circ T^{-1} = Id \implies S = T$.

Example 36
Determine all linear fractional transformations $f : \mathbb{H} \to D(0,1)$, where $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ and $D(0,1) = \{w \in si : |w| < 1\}$.
Solution:
We are looking for a transformation of the form $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$.



We are looking for linear fractional transformations of the form $f(z) = \frac{az+b}{cz+d}$ with $ad-bc \neq 0$. First of all $\mathbb{H}$ is bounded by the real axis, call it l. Thus by the Inertia principle and since we are looking for a bijective mapping, as l.f.t. are continuous, we deduce that $f(l)$ is a

circle, which is the boundary of the disk $D(0,1)$, the unit circle.
Then we consider the behaviour of the mapping at the points $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$. to find restrictions of a, b, c, d.

- $f(0) = \frac{a.0+b}{c.0+d} = \frac{b}{d}$ Since $0 \in l$, $|f(0)| = 1 \implies \left|\frac{b}{d}\right| = 1 \implies |b| = |d|$
- $f(\infty) = \frac{a}{c}$. Since $\infty \in l$, $|f(\infty)| = 1 \implies \left|\frac{a}{c}\right| = 1 \implies |a| = |c|$.
  But we have that $ad - bc \neq 0$, $a \neq 0$, and $c \neq 0$, $b \neq 0$ and $d \neq 0$. The modulus of $\frac{a}{c}$ is 1, so we can write it is complex form as $\frac{a}{c} = \cos(\theta) + i\sin(\theta) = e^{i\theta}$. Thus we have

  $$f(z) = \frac{a}{c}\frac{z + \frac{b}{a}}{z + \frac{d}{c}} = e^{i\theta}\frac{z + \frac{b}{a}}{z + \frac{d}{c}}$$

  For convenience, label $z_0 = -\frac{b}{a}$ and $z_1 = -\frac{d}{c}$ (note that this z_1 is different from the one above.) Then we have

  $$f(z) = e^{i\theta}\frac{z - z_0}{z - z_1}$$

- Since $1 \in l$, we have $|f(1)| = \left|e^{i\theta}\frac{1-z_0}{1-z_1}\right| = 1 \implies \frac{|1-z_0|}{|1-z_1|} = 1 \implies |1 - z_0| = |1 - z_1|$
  From this we can deduce the following:

  $$\begin{aligned} |1 - z_0| = |1 - z_1| &\implies |1 - z_0|^2 = |1 - z_1|^2 \implies (1 - z_0)\overline{(1 - z_0)} = (1 - z_1)\overline{(1 - z_1)} \\ &\implies (1 - z_0)(1 - \overline{z_0}) = (1 - z_1)(1 - \overline{z_1}) \implies 1 - z_0 - \overline{z_0} + |z_0|^2 = 1 - z_1 - \overline{z_1} + |z_1|^2 \\ &\implies -(z_0 + \overline{z_0}) + |z_0|^2 = -(z_1 + \overline{z_1}) + |z_1|^2 \end{aligned}$$

  But $|z_0| = \left|\frac{b}{a}\right| = \left|\frac{d}{c}\right| = |z_1| \implies |z_0|^2 = |z_1|^2$. So we get that

  $$-(z_0 + \overline{z_0}) = -(z_1 + \overline{z_1}) \implies \Re(z_0) = \Re(z_1)$$

  From here we have that $\Re(z_0) = \Re(z_1)$ and $|z_0| = |z_1| \implies z_0 = z_1$ or $z_0 = \overline{z_1}$. But $z_0 \neq z_1$, so we deduce that $z_0 = \overline{z_1}$

The final answer is thus

$$f(z) = e^{i\theta}\frac{z - z_0}{z - \overline{z_0}}$$

Chapter 6

Integration Along Curves (Contour Integration)

6.1 Curves: Definitions

Definition 40 (Parametrised Curve)
A parametrised curve is a continuous map

$$\gamma : [a, b] \to \mathbb{C}$$

given by

$$\gamma(t) = x(t) + iy(t)$$

such that the functions

$$x : [a, b] \to \mathbb{R} \text{ and } y : [a, b] \to \mathbb{R} \text{ are continuous}$$

$$\gamma(a) \text{ is the initial point and } \gamma(b) \text{ is the final/end/terminal point}$$

Definition 41
The reverse curve for $\gamma : [a, b] \to \mathbb{C}$ is $\overline{\gamma} : [a, b] \to \mathbb{C}$ such that

$$\overline{\gamma}(t) = \gamma(a + b - t)$$

$$\text{So } t \mapsto a + b - t \text{ is a linear function}$$

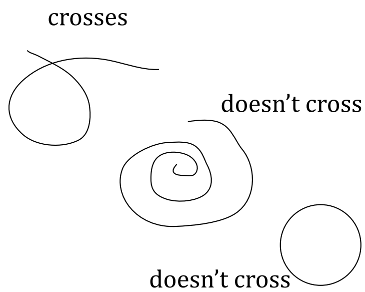
$$a \mapsto a + b - a = a \text{ and } b \mapsto a + b - b = a$$

$$\text{So } [a, b] \mapsto [a, b]$$

Definition 42 (Simple Curve)
A curve γ is called simple is its image doesn't cross itself. i.e.

$$\text{If } s < t \text{ and } \gamma(s) = \gamma(t) \text{ then } s = a, t = b$$

(so the curve may meet itself at the starting or terminal point only, e.g. like a circle)



Definition 43 (Closed Curve)
The curve γ is closed if $\gamma(a) = \gamma(b)$

Definition 44 (Simple Closed)
The curve γ is simple closed if it is simple and closed.

Definition 45 (Smooth Curve)
The curve γ is smooth if $\gamma'(t) = x'(t) + iy'(t)$ exists and is continuous.

Remark: At a and b we use one-sided derivatives

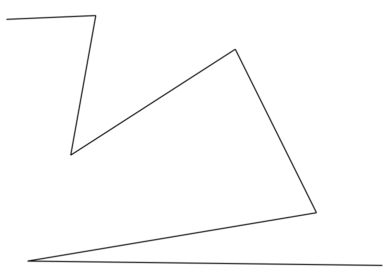
$$\gamma'(a) = \lim_{h\to 0} \frac{\gamma(a+h) - \gamma(a)}{h}$$

$$\gamma'(b) = \lim_{h\to 0} \frac{\gamma(b+h) - \gamma(b)}{h}$$

Definition 46 (Path, Piecewise Smooth)
The curve γ is a path if it is piecewise smooth, which means that γ is continuous and there exist points $a_0, a_1, a_2, \ldots, a_n$ with

$$a = a_0 < a_1 < a_2 < \ldots < a_{n-1} < a_n = b$$

such that $\gamma : [a_i, a_{i+1}] \to \mathbb{C}$ is smooth for $i = 0, 1, 2, \ldots, n-1$.
So consider the following:



The derivatives from the left and right at the corners are not the same so γ is not differentiable there. So it is not smooth. However, it is piecewise smooth.

Definition 47 (Contour)
The curve γ is a contour if it is a simple closed path.

Example 37 (Parametrising a circle)
Consider the circle with positive orientation.

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$$

It is parametrised by

$$z(t) = z_0 + r(\cos t + i \sin t) = z_0 + re^{it} \text{ where } 0 \leq t \leq 2\pi$$

The same curve with negative orientation is parametrised by

$$z(t) = z_0 + re^{-it} \text{ with } 0 \leq t \leq 2\pi$$

6.2 Important definitions and useful results for later on

Definition 48 (Length of a curve)
The length of the curve $\gamma(t) = x(t) + iy(t)$ with $a \leq t \leq b$ is given by

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_a^b |\gamma'(t)| \, dt$$

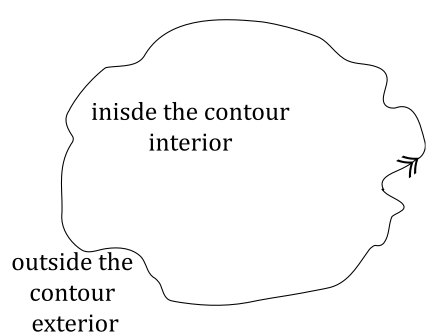
Theorem 11 (Jordan's Curve Theorem)
Let γ be a contour which is closed and simple.
Consider the complement of γ in $\mathbb{C}$, i.e.

$$\mathbb{C} - \{z : z = \gamma(t); t \in [a, b]\}$$

Then $\mathbb{C}$ is the union of two disjoint regions, Ω_1 and Ω_2, with Ω_1 bounded and Ω_2 unbounded.

$$\mathbb{C} = \gamma([a, b]) \cup \Omega_1 \cup \Omega_2$$

with Ω_1 the interior of γ and Ω_2 the exterior of γ



Definition 49

Let $F : [a,b] \to \mathbb{C}$ s.t. $F(t) = U(t) + iV(t)$ with $U, V : [a,b] \to \mathbb{R}$

Assume that U and V are piecewise smooth.

$$\int_a^b F(t)dt = \int_a^b U(t)dt + i\int_a^b V(t)dt$$

Obviously,

$$\Re\left(\int_a^b F(t)dt\right) = \int_a^b U(t)dt = \int_a^b \Re(F(t))dt$$

We have that

$$\int_a^b (F(t) + G(t))dt = \int_a^b F(t)dt + \int_a^b G(t)dt$$

If $c = \alpha + i\beta$, then

$$\int_a^b cF(t)dt = c\int_a^b F(t)dt$$

Proof. of the last part
First of all

$$cF(t) = (\alpha + i\beta)(U(t) + iV(t)) = (\alpha U(t) - \beta V(t)) + i(\alpha V(t) + \beta U(t))$$

So that

$$\int_a^b cF(t)dt = \int_a^b \alpha U(t) - \beta V(t) + i\int_a^b \alpha V(t) + \beta U(t)$$

On the other hand

$$\begin{aligned} c\int_a^b F(t)dt &= (\alpha + i\beta)\left(\int_a^b U(t)dt + i\int_a^b V(t)dt\right) \\ &= \left(\alpha\int_a^b U(t)dt - \beta\int_a^b V(t)dt\right) + i\left(\beta\int_a^b U(t)dt + \alpha\int_a^b V(t)dt\right) \\ &= \int_a^b \alpha U(t) - \beta V(t) + i\int_a^b \alpha V(t) + \beta U(t) \end{aligned}$$

We conclude that

$$c\int_a^b F(t)dt = \int_a^b cF(t)dt$$

□

Lemma 3 (Very Important)

$$\left|\int_a^b F(t)dt\right| \leq \int_a^b |F(t)|dt$$

Proof.

$$\text{Let } I = \int_a^b F(t)dt \in \mathbb{C}$$

The polar form of I is $I = |I|e^{i\theta}$ .Then we have that $|I| = Ie^{-i\theta}$. So

$$\left|\int_a^b F(t)dt\right| = |I| = Ie^{-i\theta} = e^{-i\theta}\int_a^b F(t)dt = \int_a^b e^{-i\theta}F(t)dt$$

Where the last integral is real since we took the modulus sign at the beginning. So

$$\begin{aligned}
\Re\left(\int_a^b e^{-i\theta}F(t)dt\right) &= \int_a^b e^{-i\theta}F(t)dt \\
&= \int_a^b \Re\left(e^{-i\theta}F(t)\right)dt \leq \left|\int_a^b \Re\left(e^{-i\theta}F(t)\right)dt\right| \\
&\leq \int_a^b \left|\Re\left(e^{-i\theta}F(t)\right)\right|dt = \int_a^b \left|e^{-i\theta}F(t)\right|dt \\
&= \int_a^b |F(t)|dt \text{ since } |e^{-i\theta}| = 1
\end{aligned}$$

Note: We got the last step using the fact from Analysis 2:

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

□

In the following we are assuming that:
C is a piecewise-smooth curve given by

$$z(t) = x(t) + iy(t) \text{ with } a \leq t \leq b$$

f is a function defined on an open set U containing the image of C.
$f \circ z$ is piecewise continuous on $[a, b]$

Definition 50

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Note the following

$$f(z) = u(x,y) + iv(x,y) \implies \int_C f(z)dz = \int_a^b \left[(u(x(t),y(t)) + iv(x(t),y(t))) \, . \, (x'(t) + iy'(t))\right] dt$$

$$= \int_a^b \left[u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)\right] dt$$

$$+ i \int_a^b \left[u(x(t),y(t))y'(t) + v(x(t),y(t))x'(t)\right] dt$$

$$\int_C udx - vdy + i \int_C udy + vdx$$

6.3 Properties of Line Integrals

Proposition 9
Let γ be a parametrised curve and $z(t) = x(t) + iy(t)$ with $a \le t \le b$. Denote the reverse curve by $-\gamma = z(a+b-t) = \overline{z}(t)$, with $a \le t \le b$. Then

$$\int_\gamma f(z)dz = -\int_{-\gamma} f(z)dz$$

Proof.

$$\int_{-\gamma} f(z)dz = \int_a^b f(\overline{z(t)})\overline{z'}(t)dt$$

$$= \int_a^b f(z(a+b-t))z'(a+b-t)\frac{d}{dt}(a+b-t)dt = \int_a^b f(z(a+b-t))z'(a+b-t)(-1)dt$$

Now substitute $u = a+b-t$ so that $du = -dt$ and at $t = a$, $u = b$ and $t = b$, $u = a$. So we have

$$\int_{u=b}^{u=a} f(z(u))z'(u)(-1)(-1)du = \int_b^a f(z(u))z'(u)du$$

$$= -\int_a^b f(z(u))z'(u)du = -\int_a^b f(z(t))z'(t)dt \text{ by ranaming the parameter}$$

$$= -\int_\gamma f(z)dz$$

□

Proposition 10
Let γ_1 and γ_2 be two parametrised curves with $\gamma_1(b) = \gamma_2(a)$, i.e. γ_2 follows γ_1. Then we have the curve $\gamma_1 \cup \gamma_2$, which is denoted by $\gamma_1 + \gamma_2$ and

$$\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

Proposition 11
If $|f(z)| \leq M$ on the curve γ, i.e.

$$|f(z(t))| \leq M \;\; \forall t \in [a,b] \text{ then}$$

$$\left|\int_{\gamma} f(z)dz\right| \leq ML(\gamma)$$

Proof. Consider

$$\begin{aligned}\left|\int_{\gamma} f(z)dz\right| &= \left|\int_a^b f(z(t))z'(t)dt\right| \\ &\leq \int_a^b |f(z(t))z'(t)dt| \leq \int_a^b M|z'(t)|dt \\ &= M\int_a^b |z'(t)|dt = ML(\gamma)\end{aligned}$$

□

Proposition 12 (Comparing Parametrisations)
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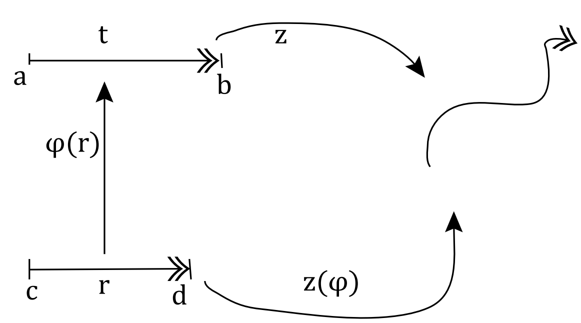


$$\phi : [c,d] \to [a,b] \text{ is bijective with continuous partial derivatives and } \phi'(r) > 0$$

$$\phi(c) = a, \phi(d) = b \text{ and } t = \phi(r) \implies dt = \phi'(r)dr$$

since ϕ is bijective and increasing, $z \circ \phi$ parametrises the curve with the same orientation. The new parametrisation is

$$z \circ \phi : [c,d] \to \mathbb{C}$$

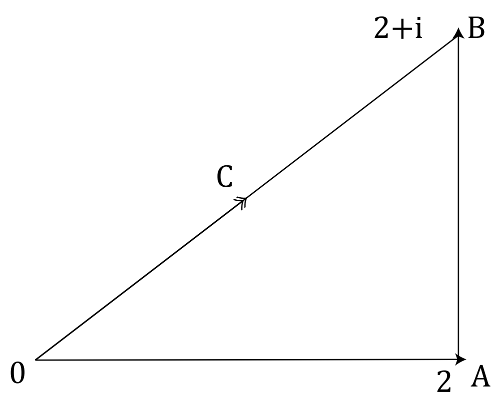
Now

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(z(t))z'(t)dt \\
&= \int_c^d f(z(\phi(r)))z'(\phi(r))\phi'(r)dr \text{ with the cahnge of variables } t = \phi(r) \\
&= \int_c^d f((z \circ \phi)(r))((z \circ \phi)'(r))dr
\end{aligned}$$

Example 38
Compute the integral

$$I = \int_C z^2 dz$$

where C is the curve



We need to give a parametrisation to the curve:
We can actually find easily the equation of C, it is $y = \frac{x}{2}$. So we can parametrise by

$$z(x) = x + i\frac{x}{2} \text{ for } 0 \le x \le 2$$

Or we can set $x = 2t$ so that $0 \le x \le 2 \implies 0 \le t \le 1$. Then $y = \frac{x}{2} = t$ and then we can parametrise by

$$z(t) = 2t + it$$

We are going to use the second parametrisation.

$$z(t) = 2t + it \implies z'(t) = 2 + i \text{ where } 0 \leq t \leq 1$$

$$f(z) = z^2 = x^2 - y^2 + 2ixy = (2t)^2 - t^2 + 2i.2t.t = (4t^2 - t^2) + i4t^2 = 3t^2 + 4it^2$$

$$f(z(t)).z'(t) = (3t^2 + 4it^2)(2 + i) = t^2(3 + 4i)(2 + i)$$

So we have that

$$\begin{aligned}
\int_C z^2 dz &= \int_0^1 t^2(3 + 4i)(2 + i)dt = (3 + 4i)(2 + i)\int_0^1 t^2 dt \\
&= \frac{1}{3}(3 + 4i)(2 + i)[t^3]_0^1 = \frac{1}{3}(3 + 4i)(2 + i) \\
&= \frac{1}{3}(6 - 4) + \frac{i}{3}(3 + 8) = \frac{2}{3} + i\frac{11}{3}
\end{aligned}$$

Now consider the same integral evaluated on the horizontal segment $[0, A]$, called γ_1 and the vertical segment $[A, B]$, called γ_2. We are going to show that we get the same result since $C = \gamma_1 + \gamma_2$ i.e. we'll show that

$$\int_C z^2 dz = \int_{\gamma_1} z^2 dz + \int_{\gamma_2} z^2 dz$$

Consider first γ_1.

$$z(x) = x + 0i = x \text{ where } 0 \leq x \leq 2$$

$$f(z(x))z'(x) = x^2.1 = x^2$$

Then

$$\int_{\gamma_1} z^2 dz = \int_0^2 x^2 dx = \frac{1}{3}2^3 = \frac{8}{3}$$

Consider now γ_2.

$$z(y) = 2 + iy \text{ where } 0 \leq y \leq 1 \text{ and } z'(y) = i$$

Then

$$\begin{aligned}
\int_{\gamma_2} z^2 dz &= \int_0^1 (2 + iy)^2.idy = \int_0^1 (4 + 4iy - y^2)idy \\
&= \int_0^1 (4i - 4y - iy^2)dy = i\left(\left[4y - \frac{y^3}{3}\right]_0^1 + \left[4i\frac{y^2}{2}\right]_0^1\right) \\
&= i\left(4 - \frac{1}{3} + 4i\frac{1}{2}\right) = i\left(\frac{11}{3} + 2i\right) = i\frac{11}{3} - 2
\end{aligned}$$

The sum of the two integrals gives us

$$\frac{8}{3} - 2 + i\frac{11}{3} = \frac{2}{3} + i\frac{11}{3}$$

which is what we got before.

Example 39 (Extremely Important Result!)
Let C be the circle $C_r(0)$ traversed anticlockwise. Then

$$\int_C z^n dz = \begin{cases} 0 \text{ if } n \neq -1 \\ 2\pi i \text{ if } n = -1 \end{cases}$$

Proof. We parametrise the circle by

$$z(t) = re^{it} \text{ with } 0 \leq t \leq 2\pi$$

So that $z'(t) = ire^{it}$.
Then $f(z(t))z'(t) = (re^{it})^n.re^{it}.i$ and so

$$\int_C f(z)dz = \int_0 2\pi(re^{it})^n re^{it} idt = \int_0^{2\pi} r^{n+1}e^{nit}e^{it}.idt$$

$$= \int_0^{2\pi} r^{n+1}e^{(n+1)it}idt = r^{n+1}i\int_0^{2\pi} e^{(n+1)it}dt$$

Case 1 $n = -1$.
Then we have that $n+1 = 0$ and

$$r^{n+1}i\int_0^{2\pi} e^{(n+1)it}dt = r^0 i\int_0^{2\pi} e^0 dt = 1.i.\int_0^{2\pi} 1 = 2\pi i$$

Case 2 $n \neq -1$.

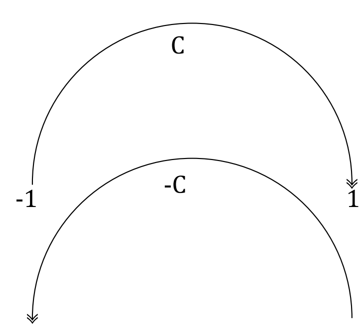
$$r^{n+1}i\int_0^{2\pi} e^{(n+1)it}dt = r^{n+1}i\frac{1}{i(n+1)}\left[e^{(n+1)it}\right]_0^{2\pi} = \frac{r^{n+1}}{(n+1)}\left(e^{2\pi i(n+1)} - e^0\right) = \frac{r^{n+1}}{(n+1)}(1-1) = 0$$

□

Example 40
Find

$$\int_C \overline{z}dz$$

over



We parametrise the curve as

$$z(t) = e^{it} \text{ for } 0 \leq t \leq \pi$$
$$z'(t) = ie^{it}$$

This is the reverse of the given curve, so we consider the integral with a - sign.

$$-\int_{-C} \overline{z} dz = -\int_0^\pi \overline{z(t)} z'(t) dt = -\int_0^\pi \left(e^{-it}\right).\left(ie^{it}\right) dt$$
$$= -\int_0^\pi i dt = -i \int_0^\pi 1 dt = -i[t]_0^\pi = -i\pi$$

Theorem 12 (Green's Theorem)
Let C be a simple closed curve bounding a region R. Let C be traversed anticlockwise. Let $P(x,y)$ and $Q(x,y)$ be continuous functions on C and R, with continuous partial derivatives. Then

$$\int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy$$

6.4 (T) Cauchy's Theorem

Theorem 13 (Cauchy-Gousat (or just Cauchy))
If f is holomorphic on a simple closed curve C and its interior, then

$$\int_C f(z) dz = 0$$

Remark: No assumptions are made on the continuity of $f'(z)$ or u_x, u_y, v_x, v_y.

Proof. Let $f(z) = u(x,y) + iv(x,y)$, where $u = \Re(f)$ and $v = \Im(f)$. Also, $dz = dx + idy$. So we have that

$$\int_C f(z) dz = \int_C (u + iv)(dx + idy) = \int_C (udx - vdy) + i(udy + vdx)$$
$$= \int_C (udx - vdy) + i \int_C (udy + vdx) \text{ to which we now apply Green's Theorem to get}$$
$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx dy + i \iint_R \left(-\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right) dx dy$$
$$= \iint_R (-v_x - u_y) dx dy + i \iint_R (-v_y + u_x) dx dy$$

But we are assuming that f is holomorphic, hence the CRE hold. So we have that

$$u_x = v_y \implies -u_y + u_x = 0$$

$$u_y = -v_x \implies -v_y - u_x = 0$$

And so the integrals become

$$\iint_R 0dxdy + i\iint_R 0dxdy = 0$$

Thus

$$\int_C f(z)dz = 0$$

$\square$

6.5 (T) The Fundamental Theorem of Line Integrals and an Important Corollary

Definition 51 (Antiderivative)
Let f be continuous on a region Ω and assume that there exists a holomorphic function F on Ω with $F'(z) = f(z)$.
Then F is called the antiderivative or primitive of f.

Theorem 14 (The Fundamental Theorem of Calculus of Line Integrals)
If f is continuous on a region Ω, F is an antiderivative of f in Ω and γ is a path joining the two points $z_1, z_2 \in \Omega$, then

$$\int_\gamma f(z)dz = F(z_2) - F(z_1)$$

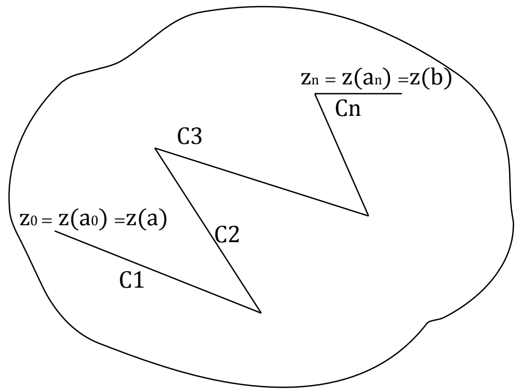
i.e. the value of the integral is independent of the path chosen and depends only on the endpoints.

Proof. Let γ be parametrised by $z(t)$ for $a \le t \le b$. Assume that γ is smooth, i,e, that $z'(t)$ is continuous.

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(z(t))z'(t)dt \text{ by definition}\\
&= \int_a^b F'(z(t))z'(t)dt\\
&= \int_a^b \frac{dF}{dt}(z(t))dt \text{ by the Chain Rule}\\
&= [F(z(t))]_a^b = F(z(a)) - F(z(b))\\
&= F(z_2) - F(z_1) \text{ since } z(a) = z_1 \text{ and } z(b) = z_2
\end{aligned}$$

$\square$

We made an assumption that the curve is smooth. What if it was piecewise smooth? It turns out that the result still holds.
Suppose the piecewise smooth curve γ consists of the curves $C_1, C_2, \ldots, C_n$ such that the curves are connected at the corners $a = a_0, a_1, a_2, \ldots, a_n = b$.



Then we have that

$$\begin{aligned}\int_\gamma f(z)dz &= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)dz \\ &= [F(z(a_1)) - F(z(a_0))] + [F(z(a_2)) - F(z(a_1))] + \ldots + [F(z(a_n)) - F(z(a_{n-1}))] \\ &= F(z(a_n)) - F(z(a_0)) \text{ since all the other terms in this telescopic sum cancel out} \\ &= F(z_2) - F(z_1)\end{aligned}$$

as before.

Corollary 7
If γ is closed inside the region Ω and f is continuous on Ω and has an antiderivative F on ω, then

$$\int_\gamma f(z)dz = 0$$

Example 41
We already showed that

$$\int_{C_r(0)} \frac{1}{z} dz = 2\pi i$$

Since the curve we are integrating is closed and we don't get 0, the integrand doesn't have an antiderivative on $\mathbb{C}/\{0\}$
But what about $Log(z)$, the principle logarithm, given by $Log(z) = \log|z| + iArg(z)$, where $-\pi < Arg(z) \leq \pi$?
This is actually not an antiderivative, since the definition of antiderivative requires it to be differentiable, and it is not even continuous everywhere. In particular, Log is not continuous on the negative real axis, and it requires a cut there.

Then what about another log? For example, consider

$$\log(z) = \log|z| + iArg(z) \text{ with } -\frac{\pi}{2} < Arg(z) \leq \frac{3\pi}{2}$$

Now this requires a cut on the negative imaginary axis, since this new log is not continuous there. In fact, not matter how we define the complex logarithm, it will always require a cut, and therefore, we cannot find an antiderivative of the function $\frac{1}{z}$ in any disk around the origin.

Example 42
Suppose we wanted to evaluate the same integral on another curve, like the curves C_1 and C_2 below. Then

$$\int_{C_1} \frac{1}{z} dz = \pi i \text{ and } \int_{C_2} \frac{1}{z} dz = -\pi i$$

So the integral depends on the curve we have chosen.
log creates discontinuities, since the argument can't be made continuous around the circle.
On the other hand, By Cauchy's Theorem,

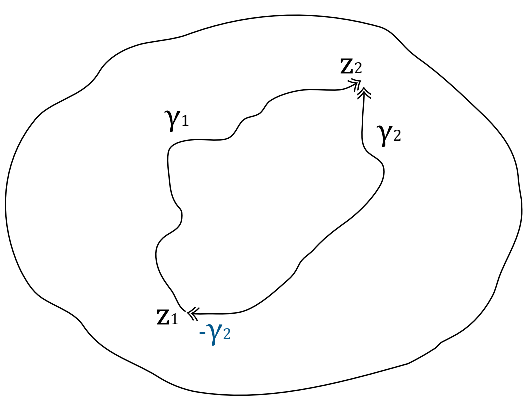
$$\int_{\gamma} f(z) dz = 0$$

So consider a similar example, but with a function which satisfies the conditions of the theorem and a curve, which is not a circle, but still closed.
we take

γ_1, γ_2 to have the same endpoint but we are traversing them in the same direction, so

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$$

Now suppose that the anticlockwise curve goes along γ_1 and comes back along γ_2. Call this curve γ. We have that



$$\gamma = \gamma_1 + (-\gamma_2)$$

And then Cauchy's Theorem gives

$$\begin{aligned} 0 = \int_\gamma f(z)dz &= \int_{\gamma_1} f(z)dz + \int_{-\gamma_2} f(z)dz \\ &= \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz \\ &= 0 \end{aligned}$$

Back to the example with $\frac{1}{z}$, we have that

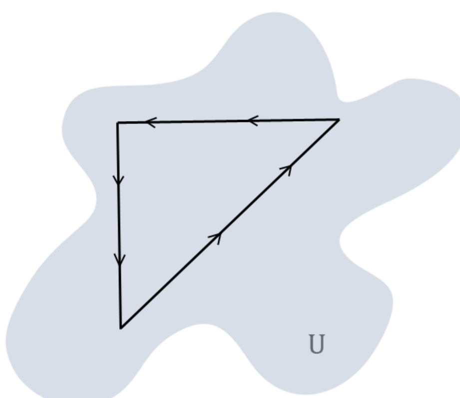
$$\gamma = C_1 + (-C_2)$$

$$\int_\gamma \frac{1}{z}dz = \int_{C_1} \frac{1}{z}dz - \int_{C_2} \frac{1}{z}dz = \pi i - (-\pi i) = 2\pi i$$

6.6 (T)Gousat's Theorem

Theorem 15 (Gousat's Theorem)
Let γ be a triangular contour with its interior contained in some domain U and let $f(z)$ be holomorphic on U. Then

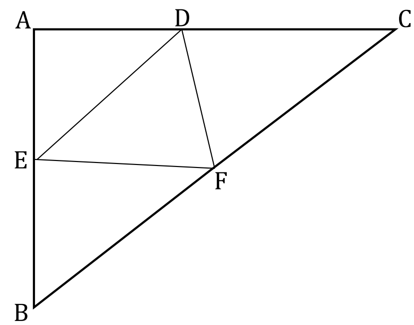
$$\int_\gamma f(z)dz = 0$$



Proof

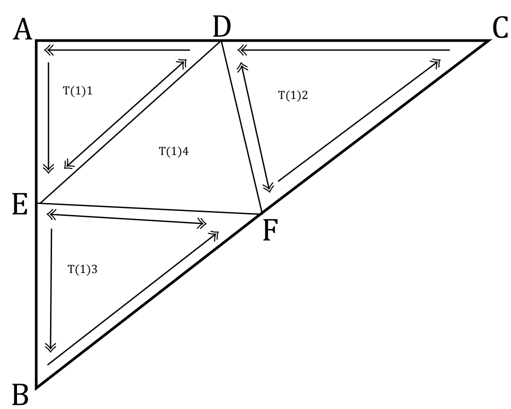
Setting

Let ABC be the triangle with contour $\gamma^{(0)}$, i.e. $\gamma^{(0)}$ is the curve ABC
Call ABC $T^{(0)}$.
Let $d^{(0)}$ be the diameter of $T^{(0)}$, i.e. the length of the longest side of ABC.
Let $p^{(0)}$ be the perimeter of ABC.
Bisect each side of the triangle ABC, such that $AD = DC$, $BF = FC$, $AE = EB$, as shown. Connect the points E, D, F to form a triangle with vertices the points E, D, F.



Then we have that
$|ED| = \frac{1}{2}|BC|$, $|DF| = \frac{1}{2}|AB|$, $|EF| = \frac{1}{2}|AC|$
Now, the lines ED, DF and EF divide the original triangle into 4 smaller triangles. Call them $T_k^{(1)}$, with $k = 1, 2, 3, 4$. We have
$T_1^{(1)}$ with boundary $\gamma_1^{(1)}$
$T_2^{(1)}$ with boundary $\gamma_2^{(1)}$
$T_3^{(1)}$ with boundary $\gamma_3^{(1)}$
$T_4^{(1)}$ with boundary $\gamma_4^{(1)}$
i.e. $T_k^{(1)}$ has boundary $\gamma_k^{(1)}$ for $k = 1, 2, 3, 4$

Step 1



We are going to find the contour integral around each of the 4 new triangles we now have.
Around ADE

$$\int_{\gamma_1^{(1)}} f(z)dz = \int_{DA} f(z)dz + \int_{AE} f(z)dz + \int_{ED} f(z)dz$$

Around CDF

$$\int_{\gamma_2^{(1)}} f(z)dz = \int_{CD} f(z)dz + \int_{DF} f(z)dz + \int_{FC} f(z)dz$$

Around BFE

$$\int_{\gamma_3^{(1)}} f(z)dz = \int_{BF} f(z)dz + \int_{FE} f(z)dz + \int_{EB} f(z)dz$$

Around DEF

$$\int_{\gamma_3^{(1)}} f(z)dz = \int_{DE} f(z)dz + \int_{EF} f(z)dz + \int_{FD} f(z)dz$$

Now, we have that

$$\int_{DE} f(z)dz = -\int_{ED} f(z)dz$$
$$\int_{DF} f(z)dz = -\int_{FD} f(z)dz$$
$$\int_{EF} f(z)dz = -\int_{FE} f(z)dz$$

And this means that

$$\begin{aligned}&\int_{\gamma_1^{(1)}} f(z)dz + \int_{\gamma_2^{(1)}} f(z)dz + \int_{\gamma_3^{(1)}} f(z)dz + \int_{\gamma_4^{(1)}} f(z)dz \\ &= \int_{CD} f(z)dz + \int_{DA} f(z)dz + \int_{AE} f(z)dz + \int_{EB} f(z)dz + \int_{BF} f(z)dz + \int_{FC} f(z)dz \\ &= \int_{\gamma^{(0)}} f(z)dz\end{aligned}$$

Our aim now is to show that this equals 0.

Step 2

We so far managed to express $\int_{\gamma^{(0)}} f(z)dz$ as a sum of 4 contour integrals, but this is not very useful. We would like to somehow compare them, so that we end up with an inequality between 2 integrals. However, we cannot compare the integrals themselves since they are complex integrals. So what we do instead is compare the moduli. We ask ourselves the question: Which $\int_{\gamma_k^{(1)}} f(z)dz$ has the biggest modulus? Since we picked an arbitrary triangle to begin with, we can't explicitly answer this question. It could be any of the 4 triangles, so we just pick some number j s.t. $1 \leq j \leq 4$ and assume that $T_j^{(1)}$ is the triangle with greatest modulus of the contour integral around $\gamma_j^{(1)}$. That is, we have

$$\max_{k=1,2,3,4} \left| \int_{\gamma_k^{(1)}} f(z)dz \right| = \left| \int_{\gamma_j^{(1)}} f(z)dz \right|$$

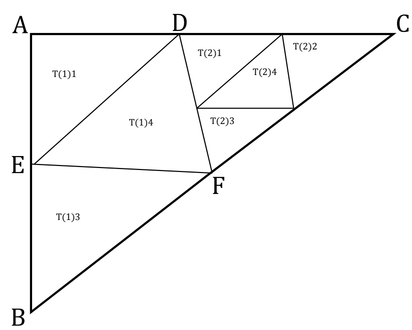
And now we can use this in the following way.

$$
\begin{aligned}
\left| \oint_{\gamma_{(0)}} f(z)dz \right| &= \left| \sum_{k=1}^{4} \oint_{\gamma_k^{(1)}} f(z)dz \right| \\
&\leq \sum_{k=1}^{4} \left| \oint_{\gamma_k^{(1)}} f(z)dz \right| \text{ (by the triangle inequality)} \\
&\leq 4 \left| \oint_{\gamma_j^{(1)}} f(z)dz \right| \text{ (*)}
\end{aligned}
$$

Step 3

Now take the triangle with subscipt j and call it $T^{(1)}$. Call its diameter $d^{(1)}$, its perimeter $p^{(1)}$ and its boundary $\gamma^{(1)}$.
We can relate the diameter and the perimeter of this triangle to the diameter and perimeter of the original triangle. Since the way we obtained the triangles was by bisecting the sides of the original triangle, we know that the largest side of the triangle $T^{(1)}$ is half the largest side of the triangle $T^{(0)}$ and similarly for the perimeter. That is, we have:
$d^{(1)} = \frac{1}{2}d^{(0)}$ and $p^{(1)} = \frac{1}{2}p^{(0)}$.

Now we concentrate only on the triangle $T^{(1)}$ and do exactly the same as with the triangle $T^{(0)}$. We bisect all the sides and connect the points which bisect the sides in order to form 4 new triangles inside of $T^{(1)}$. Call them $T_1^{(2)}$, $T_2^{(2)}$, $T_3^{(2)}$, $T_4^{(2)}$. Call their contours $\gamma_1^{(2)}$,$\gamma_2^{(2)}$,$\gamma_3^{(2)}$,$\gamma_4^{(2)}$.



Then again we choose the largest of the four new triangles, $T_j^{(2)}$, such that

$$
\left| \oint_{\gamma^{(1)}} f(z)dz \right| \leq 4 \left| \oint_{\gamma_j^{(2)}} f(z)dz \right|
$$

And now we call $T_j^{(2)}$ just $T^{(2)}$. Call its boundary $\gamma^{(2)}$, its perimeter $p^{(2)}$ and its diameter $d^{(2)}$. Then

$$p^{(2)} = \frac{1}{2}p^{(1)} = \frac{1}{4}p^{(0)}$$
$$d^{(2)} = \frac{1}{2}d^{(1)} = \frac{1}{4}d^{(0)}$$

and

$$\left| \cdot\int_{\gamma^{(1)}} f(z)dz \right| \leq 4 \left| \cdot\int_{\gamma^{(2)}} f(z)dz \right| \quad (*)$$

Now from the two (*)s, we can compare the moduli of the original contour integral and the one around $\gamma^{(2)}$. We have

$$\left| \cdot\int_{\gamma^{(0)}} f(z)dz \right| \leq 4^2 \left| \cdot\int_{\gamma^{(2)}} f(z)dz \right|$$

Step 4

Keep repeating the process. We once again divide the biggest new triangle into 4 smaller triangles and compare their perimeters, diameters and moduli of the contour integrals around their boundaries. We end up with

$$d^{(n)} = \frac{1}{2^n}d^{(0)}$$

$$p^{(n)} = \frac{1}{2^n}p^{(0)}$$

and, most importantly,

$$\left| \cdot\int_{\gamma^{(0)}} f(z)dz \right| \leq 4^n \left| \cdot\int_{\gamma^{(n)}} f(z)dz \right|$$

Step 5

We'll show that the LHS of the above inequality tends to 0 and from there reach the statement of the theorem.

Now we notice that we kept taking triangles that are within bigger triangles, such that the smaller triangles are always contained in the bigger triangles. We can make an analogy with nested intervals, where the triangles are our "intervals". This implies that there exists a unique point z_0 which is contained in all triangles $T^{(n)}$.

The point obviously also belongs to the original triangle $T^{(0)}$ and $f(z)$ is holomorphic at z_0. So the derivative at z_0 of $f(z)$ exists and we have

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
$$\iff \lim_{z\to z_0} \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0} = 0$$

Call the numerator of the second fraction ψ, i.e. we define

$$\psi(z) = f(z) - f(z_0) - f'(z_0)(z - z_0)$$

so that $\lim\limits_{z\to z_0} \frac{\psi(z)}{z-z_0} = 0$.
Now, holomorphic $\implies$ continuous, so we can use the ϵ, δ definition of continuity at a point to get

$$\forall\epsilon > 0\ ,\ \exists\delta > 0 \text{ s.t. } 0 < |z - z_0| < \delta \implies \left|\frac{\psi(z)}{z - z_0}\right| < \epsilon$$

Choose n large enough so that $d^{(n)} < \delta$, so $0 < |z - z_0| \le d^{(n)} < \delta$ (**)

Step 6

We're interested in $\left|\int_{\gamma^{(n)}} f(z)dz\right|$, so it makes sense to want to look at $\int_{\gamma^{(n)}} f(z)dz$.
Now

$$\psi(z) = f(z) - f(z_0) - f'(z_0)(z - z_0) \iff f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)$$

Integrating both sides, we get

$$\int_{\gamma^{(n)}} f(z)dz = \int_{\gamma^{(n)}} f(z_0)dz + \int_{\gamma^{(n)}} f'(z_0)(z - z_0)dz + \int_{\gamma^{(n)}} \psi(z)dz$$

Since $\int_{\gamma^{(n)}} f(z_0)dz$ has f continuous and has an antiderivative and we're integrating around a closed curve, it gives is just 0.
Similarly for $\int_{\gamma^{(n)}} f'(z_0)(z - z_0)dz$. Its antidervative is $f'(z_0)\frac{1}{2}(z - z_0)^2$. So this integral also gives us 0.

Thus, we end up with

$$\int_{\gamma^{(n)}} f(z)dz = \int_{\gamma^{(n)}} \psi(z)dz$$

Now we take the moduli of both sides (and the equality still holds)

$$\left|\int_{\gamma^{(n)}} f(z)dz\right| = \left|\int_{\gamma^{(n)}} \psi(z)dz\right|$$

But we also have that

$$\begin{aligned}
\left| . \int_{\gamma^{(n)}} \psi(z)dz \right| &\leq \max_{z\epsilon\gamma^{(n)}} |\psi(z)|.L(\gamma^{(n)}) \\
&= \max_{z\epsilon\gamma^{(n)}} |\psi(z)|.p^{(n)} \text{ (since the length of the triangualar curve is its perimeter)} \\
&\leq \epsilon. \max_{z\epsilon\gamma^{(z)}} |z - z_0|.p^{(n)} \text{ (follows from (**) above)} \\
&\leq \epsilon.d^{(n)}.p^{(n)} \text{ (again look at (**))} \\
&= \epsilon.\frac{1}{2^n}d^{(0)}.\frac{1}{2^n}p^{(0)}
\end{aligned}$$

And so we have that

$$\left| . \int_{\gamma^{(n)}} f(z)dz \right| \leq \epsilon.\frac{1}{2^n}d^{(0)}.\frac{1}{2^n}p^{(0)}$$

which in turn means that

$$\left| . \int_{\gamma^{(0)}} f(z)dz \right| \leq 4^n \left(\epsilon.\frac{1}{2^n}d^{(0)}.\frac{1}{2^n}p^{(0)} \right)$$

or,

$$\left| . \int_{\gamma^{(0)}} f(z)dz \right| \leq \epsilon.d^{(0)}.p^{(0)}$$

The diameter and perimeter are positive real numbers and this is inequality is true for any $\epsilon > 0$. So the RHS gets arbitrarily close to 0, so it must be true that

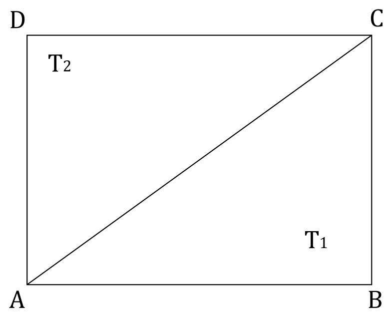
$$\int_{\gamma^{(0)}} f(z)dz = 0$$

QED

Example 43 (Application of Gousat's Theorem)
Let f be holomorphic on the rectangular path R and its interior. Then

$$\int_R f(z)dz = 0$$

Proof. We split the rectangle into 2 triangles as shown:



We know from Gousat's Theorem that

$$\int_{T_1} f(z)dz = 0 \text{ and } \int_{T_2} f(z)dz = 0 \implies \int_{T_1+T_2} f(z)dz = 0$$

Now, we have that

$$\int_{T_1} f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CA} f(z)dz$$

and

$$\int_{T_2} f(z)dz = \int_{CD} f(z)dz + \int_{DA} f(z)dz + \int_{AC} f(z)dz$$

So

$$\begin{aligned}\int_{T_1+T_2} f(z)dz = 0 &\implies \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CA} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz + \int_{AC} f(z)dz = 0\\ &\implies \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CA} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz - \int_{CA} f(z)dz = 0\\ &\implies \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz = 0\\ &\implies \int_{R} f(z)dz = 0\end{aligned}$$

□

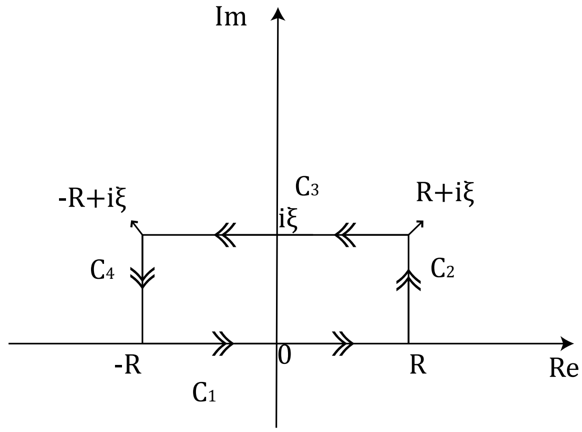
Example 44 (The Fourier Transform of the Gaussian)
Prove that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x} dx = e^{-\pi \xi^2}$$

Proof. First of all, we notice that if we take $\xi = 0$, we get the standard Gaussian

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = e^{-\pi 0^2} = e^0 = 1$$

Now let $\xi > 0$. We are going to use the following contour



Call the whole curve R. Then $R = C_1 + C_2 - C_3 - C_4$, where $-C_3$ and $-C_4$ are the curves C_3 and C_4 but with reversed orientation, so that the curve R is in the positive direction overall.
The parametrisations along C_1, C_2, C_3, C_4 are as follows:

$C_1:$ Take $z(t) = t$ for $-R \le t \le R$. and $z'(t) = 1$
$C_2:$ Take $z(t) = R + it$ for $0 \le t \le \xi$. and $z'(t) = i$
$C_3:$ Take $z(t) = i\xi - t$ for $-R \le t \le R$. but we are going to use the reverse (positive) of C_3, so take $-C_3$ by $z(t) = i\xi + t$ for $-R \le t \le R$. and $z'(t) = 1$
$C_4:$ Take $z(t) = -R - it$ for $0 \le t \le \xi$, but we take the reverse curve $-C_4$ by $z(t) = -R + it$ for $0 \le t \le \xi$. and $z'(t) = i$
The function we are going to consider along this contour is

$$f(z) = e^{-\pi z^2}$$

This function is holomorphic on R and its interior, so we can apply Cauchy's Theorem:

$$\int_R f(z)dz = 0 \implies 0 = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz$$

We are going to consider each of the integrals and we are going to let the sides of the rectangle grow to ∞, thus making each line segment infinite.
Part 1 C_1
We have that

$$f(z(t))z'(t) = f(t).1 = e^{-\pi t^2}$$

$$\int_{C_1} f(z)dz = \int_{-R}^{R} e^{-\pi t^2} dt \to \int_{-\infty}^{\infty} e^{-\pi t^2} dt \text{ as } R \to \infty$$

But $\int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1$, as we already showed. So the contribution from this integral as $R \to \infty$ is 1.

Part 2 C_2
Here

$$f(z(t))z'(t) = f(R+it).i = e^{-\pi(R+it)^2}.i$$

$$\int_{C_2} f(z)dz = \int_0^{\xi} e^{-\pi(R+it)^2} i dt = \int_0^{\xi} ie^{-\pi(R^2-t^2+2iRt)}dt$$

Consider the modulus of this integral. We have

$$\left|\int_{C_2} f(z)dz\right| \leq \int_0^{\xi} \left|ie^{-\pi(R^2-t^2+2iRt)}\right| dt = \int_0^{\xi} \left|e^{-\pi R^2}\right|\left|e^{\pi t^2}\right|\left|e^{-2\pi iRt}\right| dt$$

$$= \int_0^{\xi} e^{-\pi R^2} e^{\pi t^2} dt = e^{-\pi R^2} \int_0^{\xi} e^{\pi t^2} dt$$

The last integral is something we can't evaluate but it turns out we don't even need to, because we are only interested in what happens as we let $R \to \infty$. Doing this doesn't change the integral, but $e^{-\pi R^2} \to 0$ as $R \to \infty$.
Thus

$$e^{-\pi R^2} \int_0^{\xi} e^{\pi t^2} dt \to 0 \text{ as } R \to \infty$$

So we conclude (by the Sandwich Theorem) that

$$\int_{C_2} f(z)dz = 0 \text{ as } R \to \infty$$

So this integral gives no contribution.
Part 3 C_3.
Since we are evaluating the whole integral over R in the clockwise direction, we take

$$\int_{C_3} f(z)dz = -\int_{-C_3} f(z)dz$$

So with the positive orientation, we have:

$$f(z(t))z'(t) = f(i\xi + t).1 = e^{-\pi(t+i\xi)^2}$$

$$-\int_{-C_3} f(z)dz = -\int_{-R}^{R} e^{-\pi(t+i\xi)^2} dt = -\int_{-R}^{R} e^{-\pi(t^2-\xi^2+2it\xi)}dt$$

$$= -\int_{-R}^{R} e^{-\pi t^2} e^{\pi\xi^2} e^{-2\pi i\xi t} = -e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi t^2 - 2\pi i\xi t}dt$$

$$= -e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i\xi x}dx$$

$$\to -e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i\xi x}dx \text{ as } R \to \infty$$

Part 4 C_4
Similar calculations to the ones for C_2 show that the contribution from this integral as $R \to \infty$ is 0.
Part 5 Combining all together:

$$0 = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz$$

We let $R \to \infty$ and we get that

$$0 = 1 + 0 - e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i\xi x} dx + 0$$

So we have that

$$0 = 1 - e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i\xi x} dx$$

$$\implies e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i\xi x} dx = 1$$

$$\implies \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i\xi x} dx = e^{-\pi\xi^2}$$

This is what we wanted to prove. Note that we can take real and imaginary parts to get that:

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} \cos(2\pi\xi x) dx$$

$$0 = \int_{-\infty}^{\infty} e^{-\pi x^2} \sin(2\pi\xi x) dx$$

□

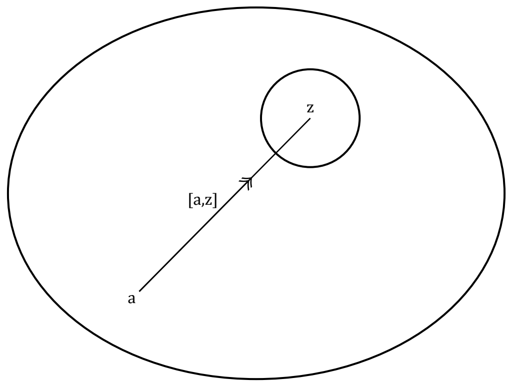
6.7 (T) The Antiderivative Theorem

The following theorem tells us when a function has an antiderivative.

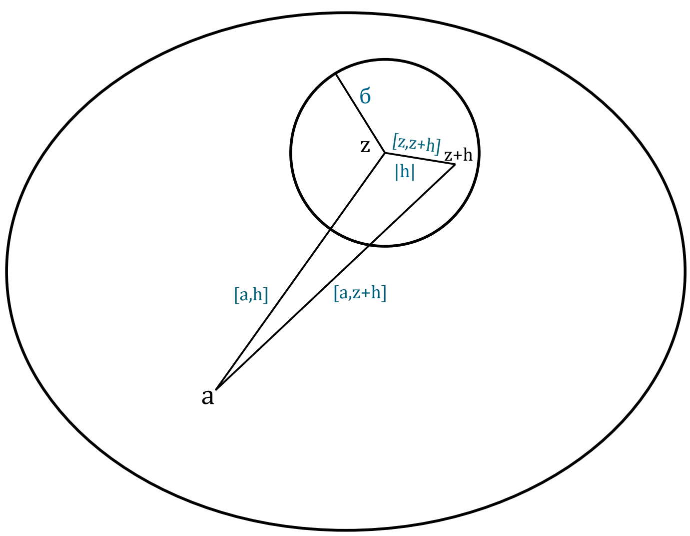
Theorem 16 (The antiderivative theorem)
Let Ω be a convex region. Let $f(z)$ be continuous on Ω with $\int_\gamma f(z)dz = 0$, $\forall \gamma$ triangular contours contained in Ω. (note that we proved this for $f(z)$ holomorphic.
Fix $a \in \Omega$ and define

$$F(z) = \int_{[a,z]} f(w)dw$$

Then F is a primitive of $f(z)$ on Ω (i.e. $F'(z) = f(z), \forall z \in \Omega$)



Proof. Fix $z \in \Omega$ and $\delta > 0$ s.t. $D(z,\delta) \subset \Omega$.
Take $h \in \mathbb{C}$ s.t. $|h| < \delta$ so that $z+h \in D(z,\delta) \subset \Omega$
Now, let a be a point in Ω which is outside of the disk we just constructed. Connect a to the points z and $z+h$ to form a triangle like this one:



Now we use Gausat's Theorem on the triangle with vertices a, z, $z+h$. So he have:

$$\int_{[a,z]} f(w)dw + \int_{[z,z+h]} f(w)dw + \int_{[z+h,a]} f(w)dw = 0$$
$$= \int_{[a,z]} f(w)dw + \int_{[z,z+h]} f(w)dw - \int_{[a,z+h]} f(w)dw$$

Here we recognise

$$\int_{[a,z]} f(w)dw = F(z)$$

and

$$\int_{[a,z+h]} f(w)dw = F(z+h)$$

by the statement of the theorem. So we have:

$$F(z) + \int_{[z,z+h]} f(w)dw - F(z+h) = 0$$

Now, our aim is to show that

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

So we will work with

$$F(z+h) - F(z) = \int_{[z,z+h]} f(w)dw$$

to show that.
Consider

$$\begin{aligned}
\frac{F(z+h) - F(z)}{h} - f(z) &= \frac{1}{h}\int_{[z,z+h]} f(w)dw - f(z) \\
&= \frac{1}{h}\int_{[z,z+h]} f(w)dw - \frac{1}{h}\int_{[z,z+h]} f(z)dw \\
&\text{(Here we used a trick - we represented } f(z) \text{ as an integral)} \\
&= \frac{1}{h}\int_{[z,z+h]} (f(w) - f(z))\, dw
\end{aligned}$$

Let's consider now the modulus

$$\begin{aligned}
\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \frac{1}{|h|}\left|.\int_{[z,z+h]} (f(w) - f(z))\, dw\right| \\
&\le \frac{1}{|h|}.|h|. \max_{w\in[z,z+h]} |f(w) - f(z)|\text{ , where } |h| \text{ is the length of the line segment} \\
&\le \max_{w\in[z,z+h]} |f(w) - f(z)|
\end{aligned}$$

Now, since f is continuous at z, we have, by the ϵ, δ definition of continuity at a point,

$$\forall\epsilon > 0, \exists\delta_1 > 0 \text{ s.t. } |w - z| < \delta_1 \implies |f(w) - f(z)| < \epsilon$$

By making the disk as small as possible, we can take $h < \delta_1$ so that

$$\max_{w\in[z,z+h]} |f(w) - f(z)| < \epsilon$$

and so it tends to 0 as $\epsilon \to 0$, which proves that

$$\frac{F(z+h) - F(z)}{h} \to f(z) \text{ as } \epsilon \to 0$$

□

Corollary 8
On an open disk a holomorphic function f has a primitive (the disk is a convex region) $\implies$ if γ is a closed curve inside the disk,

$$\int_{\gamma} f(z)dz = 0$$

Corollary 9
Let f be holomorphic on an open set U containing a circle C and its interior. Then

$$\int_{C} f(z)dz = 0$$

sketch proof. Let C' be a circle of radius slightly bigger than that of C. Then we can apply the previous corollary to the disk C bounded by C'. □

Example 45
γ is any closed curve in $\mathbb{C}$. Then

$$\int_{\gamma} e^z dz = 0$$

since e^z is holomorphic (entire)everywhere in $\mathbb{C}$

Example 46
For $n \in \mathbb{N}$,

$$\int_{\gamma} z^n dz = 0$$

Note: We proved this result before when considering the general case for all n.

Example 47
γ is the circle $|z| = 2$ traversed anticlockwise. Then

$$\int_{\gamma} \frac{\cos(z).\cosh(3z+\pi)}{(z^2+16)(z^3-28)} dz = 0$$

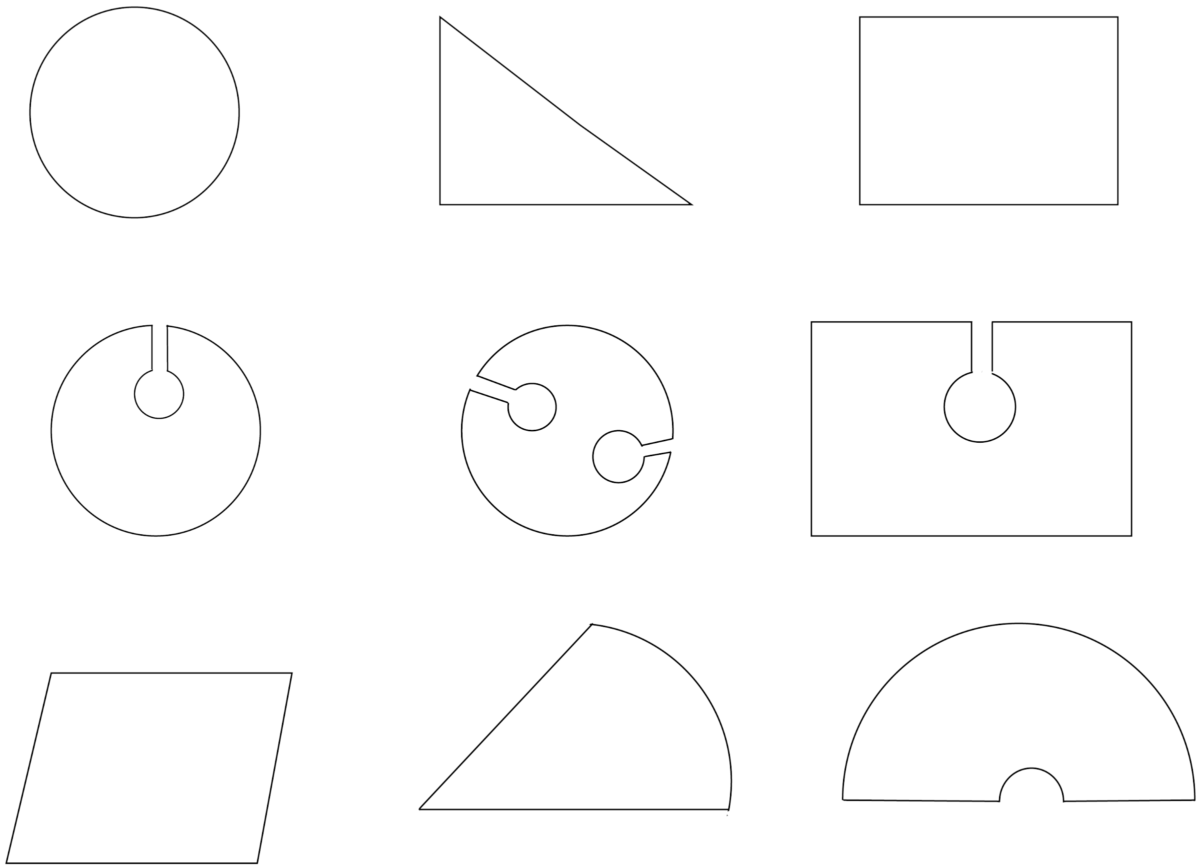
We notice that there are possible discontinuities arising from potential 0's in the denominator. However, these 0's don't occur in the region we are working over.

$$z^2 + 16 = 0 \iff z = + - 4i$$
$$z^3 - 28 = 0 \iff z = \sqrt[3]{28}, \sqrt[3]{28}e^{\frac{2\pi i}{4}}, \sqrt[3]{28}e^{\frac{4\pi i}{3}}$$

All of these points are outside the disk of radius 2.

6.8 Toy contours

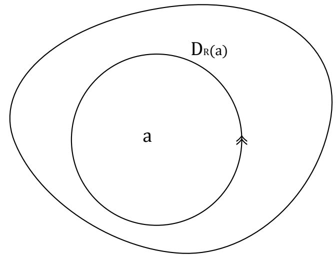
The following are the kinds of contours we are going to work with. They are called, in order:
circle (or semi-circle), triangle, rectangle, keyhole (circle), keyhole (rectangle), parallelogram, sector, indented region



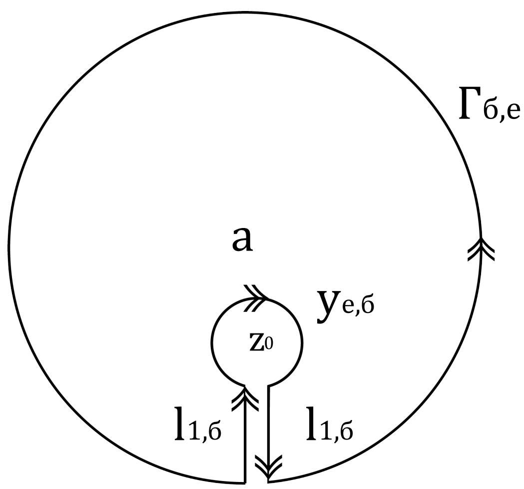
6.9 (T)Cauchy's Integral Formula

Theorem 17 (Cauchy's Integral Formula)
Let f be holomorphic on an open set containing a closed disk with boundary the circle $C = C_R(a)$ traversed anticlockwise. Let $z_0 \in C$. Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$



Proof. We are going to work with the following keyhole contour $\Gamma_{\delta,\varepsilon}$:
The big almost-circle is C_δ, traversed anticlockwise. It is missing an infinitesimal part of its perimeter of length δ. The full circle is labelled C.
The small almost-circle is $\gamma_{\varepsilon,\delta}$, traversed clockwise. It is centred at z_0 and has radius ε. It is missing a part of its perimeter of infinitesimal length δ. The full circle is labelled γ_ε
The line segment $l_{1\delta}$ is the one going inwards from the big circle to the small circle.
The line segment $l_{2\delta}$ is the one going outwards from the small circle to the big circle.
δ is the width of the corridor.



Now, $F(z) = \frac{f(z)}{z-z_0}$ is holomorphic inside $\Gamma_{\delta,\varepsilon}$, and so, by Cauchy's Theorem,

$$\int_{\Gamma_{\delta,\varepsilon}} F(z)dz = 0$$

We now evaluate the same integral by splitting the contour into the 4 constituent parts listed above and we get that

$$0 = \int_{C_\delta} F(z)dz + \int_{l_{1,\delta}} F(z)dz + \int_{l_{2,\delta}} F(z)dz + \int_{\gamma_{\varepsilon,\delta}} F(z)dz$$

Let's consider first the integrals along the two vertical line segments. We let $\delta \to 0$, and the two segments become essentially the same segment. But since the original segments had different directions, the two integrals sum to 0:

$$\int_{l_{1,\delta}} F(z)dz + \int_{l_{2,\delta}} F(z)dz \to 0 \text{ as } \delta \to 0$$

Similarly, we consider the integral along C_δ as $\delta \to 0$, the circle becomes a closed circle. Same for the integral around $\gamma_{\varepsilon,\delta}$ which become γ_ε. That is, we have:

$$\int_{C_\delta} F(z)dz \to \int_C F(z)dz \text{ as } \delta \to 0$$

$$\int_{\gamma_{\varepsilon,\delta}} F(z)dz \to \int_{\gamma_\varepsilon} F(z)dz \text{ as } \delta \to 0$$

So overall, as $\delta \to 0$, we end up with

$$0 = \int_C \frac{f(z)}{z - z_0}dz + \int_{\gamma_\epsilon} \frac{f(z)}{z - z_0}dz$$

The integral around C is the one we want to still have in the end, but we need to manipulate the other integral to get the $f(z_0)$ in the formula. Now, we do the following trick:

$$\int_{\gamma_\varepsilon} \frac{f(z)}{z - z_0}dz = \int_{\gamma_\varepsilon} \frac{f(z) - f(z_0) + f(z_0)}{z - z_0}dz = \int_{\gamma_\varepsilon} \frac{f(z) - f(z_0)}{z - z_0}dz + \int_{\gamma_\varepsilon} \frac{f(z_0)}{z - z_0}dz$$

We are going to consider these two integral individually and show that one of them gives us the LHS of the formula, while the other one vanishes.
Consider first the second integral.
$f(z_0)$ is a constant so we can write

$$\int_{\gamma_\varepsilon} \frac{f(z_0)}{z - z_0}dz = f(z_0)\int_{\gamma_\varepsilon} \frac{1}{z - z_0}dz$$

And we can evaluate this integral by parametrising with $z(t) = z_0 + \varepsilon e^{it}$, $0 \le t \le 2\pi$, so that $dz = i\varepsilon e^{it}$ and

$$\begin{aligned}\int_{\gamma_\varepsilon} \frac{f(z_0)}{z - z_0}dz &= -f(z_0)\int_0^{2\pi} \frac{1}{\varepsilon e^{it}} i\varepsilon e^{it}dt \\ &= -f(z_0)\int_0^{2\pi} idt \\ &= -2\pi i f(z_0)\end{aligned}$$

Now consider the integral

$$\int_{\gamma_\varepsilon} \frac{f(z) - f(z_0)}{z - z_0} dz$$

We recognise something in the integrand. It is that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

Note that the derivative exists since the function is holomorphic.
Now this implies that

$$\forall E > 0, \exists \Delta > 0 \text{ s.t. } 0 < |z - z_0| < \Delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < E$$

We can make the following choices: Let $E = 1$, fix $\Delta > 0$ and take $\varepsilon < \Delta$.
We now consider the modulus of the integral with the intention to show that it is zero.

$$\begin{aligned}
\left| \cdot \int_{\gamma_\varepsilon} \frac{f(z) - f(z_0)}{z - z_0} dz \right| &= \left| \int_{\gamma_\varepsilon} \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) + f'(z_0) dz \right| \\
&\leq 2\pi\varepsilon. \max_{|z - z_0| = \varepsilon} \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) + f'(z_0) \right| \\
&\leq 2\pi\varepsilon \max_{|z - z_0| = \varepsilon} \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| + |f'(z_0)| \\
&< 2\pi\varepsilon \left(1 + |f'(z_0)|\right) \\
&\to 0 \text{ as } \varepsilon \to 0
\end{aligned}$$

So, combining all together, we have

$$0 = \int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \implies f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

□

Example 48
Let C be the circle $|z| = 2$. Evaluate, using Cauchy's Integral Formula,

$$\int_C \frac{z}{(9 - z^2)(z + i)} dz$$

Let $z_0 = -i$ and $f(z) = \frac{z}{9 - z^2}$. Then

$$\begin{aligned}
\int_C \frac{f(z)}{z + i} dz &= 2\pi i f(-i) \\
&= 2\pi i \frac{-i}{9 - (-i)^2} \quad = \frac{2\pi i(-i)}{9 - (-1)} = \frac{\pi}{5}
\end{aligned}$$

Example 49
With the same circle, consider

$$\int_C \frac{\cos z}{z^2+1} dz$$

Failed attempt:

$$\int_C \frac{\cos z}{z^2+1} dz = \int_C \frac{\cos z}{(z+i)(z-i)} dz$$

so take $f(z) = \frac{\cos z}{z-i}$ and $z_0 = -i$. This won't work since f is not a holomorphic function inside the circle of radius 2, since it has a singularity at i.
Correct attempt: By using partial fraction, we rewrite the integral as

$$\int_C \frac{\cos z}{z^2+1} dz = \frac{-1}{2i} \int_C \frac{\cos z}{z+i} dz + \frac{1}{2i} \int_C \frac{\cos z}{z-i} dz$$

Now we can apply Cauchy's Integral Formula with $f(z) = \cos z$, to get

$$\begin{aligned}
&= \frac{2\pi i}{2i} \cos(-i) + \frac{2\pi i}{2i} \cos(i) \\
&= -\pi \cos(-i) + \pi \cos(i) \\
&= -\pi \frac{e^{-i.i} + e^{i.i}}{2} + \pi \frac{e^{i.i} + e^{-i.i}}{2} \\
&= 0
\end{aligned}$$

6.10 (T) Cauchy's Formula for f' and higher derivatives

Theorem 18 (Cauchy's Formula for f')

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz$$

Proof. We start with

$$\frac{f(z_0+h) - f(z_0)}{h}$$

with the intention to take te limit as $h \to \infty$ later.

We use Cauchy's Integral Formula to write this as

$$\begin{aligned}
&\frac{f(z_0+h)-f(z_0)}{h} \\
&= \frac{\left[\frac{1}{2i\pi}\int\limits_C \frac{f(z)}{z-(z_0+h)}dz - \frac{1}{2i\pi}\int\limits_C \frac{f(z)}{z-z_0}dz\right]}{h} \\
&= \frac{1}{2i\pi h}\int\limits_C \left(\frac{f(z)}{z-(z_0+h)} - \frac{f(z)}{z-z_0}\right)dz \\
&= \frac{1}{2i\pi h}\int\limits_C f(z)\left(\frac{z-z_0-z+(z_0+h)}{(z-(z_0+h))(z-z_0)}\right)dz \\
&= \frac{1}{2i\pi}\int\limits_C \frac{f(z)}{(z-z_0)(z-(z_0+h))}dz
\end{aligned}$$

We would like to show that when $h \to 0$,

$$\frac{1}{2i\pi}\int\limits_C \frac{f(z)}{(z-z_0)(z-(z_0+h))}dz \to \frac{1}{2i\pi}\int\limits_C \frac{f(z)}{(z-z_0)(z-z_0)}dz$$

If we can show this, then we have proved the theorem.
So now we need to prove:

$$\lim_{h\to 0}\int\limits_C \frac{f(z)}{(z-z_0-h)(z-z_0)}dz = \int\limits_C \frac{f(z)}{(z-z_0)^2}dz$$

or,

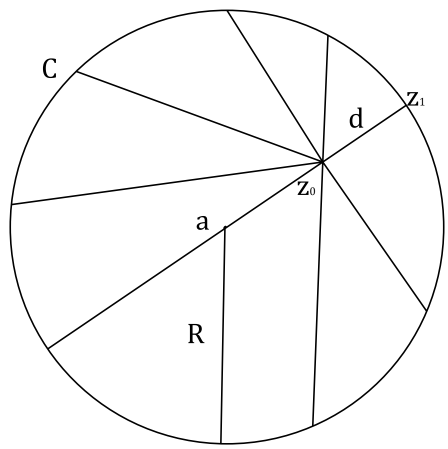
$$\int\limits_C \frac{f(z)}{(z-z_0-h)(z-z_0)}dz - \int\limits_C \frac{f(z)}{(z-z_0)^2}dz \to 0$$

Once again, we will use the technique of looking at the modulus of the integral when we don't know what to say about the integral itself. So we consider

$$\begin{aligned}
\left|\int\limits_C \frac{f(z)}{(z-z_0-h)(z-z_0)}dz - \int\limits_C \frac{f(z)}{(z-z_0)^2}dz\right| &= \left|\int\limits_C \left[\frac{f(z)}{(z-z_0)}\left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right)\right]dz\right| \\
&= \left|\int\limits_C \left[\frac{f(z)}{z-z_0}\left(\frac{z-z_0-z+z_0+h}{(z-z_0-h)(z-z_0)}\right)\right]dz\right| \\
&= \left|\int\limits_C \left[\frac{f(z)}{(z-z_0)^2}\left(\frac{h}{(z-z_0-h)}\right)\right]dz\right| \\
&\le (2\pi R).\max_{|z-a|=R}\left|\frac{f(z)h}{(z-z_0)^2(z-z_0-h)}\right|
\end{aligned}$$

where $2\pi R$ is the perimeter of the circle of radius R and a is

Now we only need to consider $\left|\frac{f(z)h}{(z-z_0)^2(z-z_0-h)}\right|$. For this, we have the following diagram:



where a is the center of the circle we are considering, R is the radius and z_0 is a point inside the disk, z_1 is a point on the circle. We're interested in the distance $|z - z_0|$. for any z on the circle. Now, the shortest distance from a fixed z_0 to any point z_1 on the circle is along the diagonal through a, and this distance is $d = |z_1 - z_0|$. We can drop the subscript 1, so that $d = |z_1 - z_0| = \min|z - z_0|$ (as we chose z_1 to be the point closest to z_0). So now we have:

$$\begin{aligned}\left|\int_C \frac{f(z)}{(z-z_0-h)(z-z_0)}dz - \int_C \frac{f(z)}{(z-z_0)^2}dz\right| &\le (2\pi R).\max_{|z-a|=R}\left|\frac{f(z)h}{(z-z_0)^2(z-z_0-h)}\right| \\ &\le (2\pi R).\max_{z\in C}\frac{|f(z)h|}{d^2.|z-z_0-h|} \\ &\le (2\pi R).\max_{z\in C}\frac{|f(z)h|}{d^2.(d-|h|)} \\ &\text{since } |z-z_0-h| \ge |z-z_0| - |h| \ge d - |h| \\ &\to 0 \text{ as } h \to 0\end{aligned}$$

□

Theorem 19 (Cauchy's Formula for the n^{th} derivative)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

See proof in homework.

Example 50
Show that

$$\int_0^\infty \frac{1-\cos(x)}{x^2}dx = \frac{\pi}{2}$$

This integral has nothing to do with complex numbers at first glance nad it takes some experience to be able to notice the complex function and contour to use to arrive at the

result.
Here we will consider
$$f(z) = \frac{1 - e^{iz}}{z^2}$$
and we are going to integrate over the contour i.e. the upper semi-circle of radius R, but with a small semi-circular bit cut off from the contour very close to 0, i.e. a distance ε from 0.
Our contour γ consists of the following:
The curve γ_R^+, i.e. the big semi-circle traversed anti-clockwise from $-R$ to R.
The line segment $[-R, -\varepsilon]$, traversed from left to right
The curve γ_ε, i.e. the small semi-circle traversed clockwise from $-\varepsilon$ to ε
The line segment $[\varepsilon, R]$, traversed from left to right

Now, we have chosen $f(z) = \frac{1-e^{iz}}{z^2}$, which is a holomorphic function and hence, by Cauchy's Theorem, we have
$$\int_\gamma f(z)dz = 0$$
So we can write
$$\begin{aligned} 0 &= \int_\gamma f(z)dz \\ &= \int_{\gamma_R^+} f(z)dz + \int_{[-R,-\varepsilon]} f(z)dz + \int_{\gamma_\varepsilon} f(z)dz + \int_{[\varepsilon,R]} f(z)dz \end{aligned}$$
Now we are going to consider each term individually.

Term 1
$$\int_{\gamma_{[-R,-\varepsilon]}} f(z)dz$$
We can parametrise the curve γ_R^+ with $z(x) = x$, $-R \le x \le -\varepsilon$
So $z'(x) = 1$ and $f(z(x)) = f(x)//$ Thus
$$\int_{\gamma_{[-R,-\varepsilon]}} \frac{1-e^{iz}}{z^2} = \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2} dx$$
Now let's substitute $x = -u$ (for reasons which will become clear later), so that $du = -dx$ and at $x = -R$ we have $u = R$ and at $x = -\varepsilon$ we have $u = \varepsilon$. And so we have the integral
$$\int_R^\varepsilon \frac{1-e^{-iu}}{-u^2} du = -\int_R^\varepsilon \frac{1-e^{iu}}{u^2} du = \int_\varepsilon^R \frac{1-e^{-iu}}{u^2} du$$

And now just replace u with x again to get

$$\int_{\varepsilon}^{R} \frac{1-e^{-ix}}{x^2} dx$$

Term 2

$$\int_{[\varepsilon,R]} f(z) dz$$

We can parametrise the curve $[\varepsilon, R]$ with $z(x) = x$, $\varepsilon \le x \le R$
So $z'(x) = 1$ and $f(z(x)) = f(x)$
So we get

$$\int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2} dx$$

Now the sum of **Term 1** and **Term 2** is

$$\begin{aligned}
\int_{\varepsilon}^{R} \frac{1-e^{-ix}}{x^2} dx + \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2} dx &= \int_{\varepsilon}^{R} \frac{1-e^{-ix}+1-e^{ix}}{x^2} dx \\
&= \int_{\varepsilon}^{R} \frac{2-(e^{ix}+e^{-ix})}{x^2} dx \\
&= \int_{\varepsilon}^{R} \frac{2-2(\cos x)}{x^2} dx \\
&= 2\int_{\varepsilon}^{R} \frac{1-cosx}{x^2} dx
\end{aligned}$$

Now as we increase the size of the bigger circle, i.e. we let $R \to \infty$ and decrease the size of the smaller circle. i.e. we let $\varepsilon \to 0$, we effectively are trying to cover the whole upper complex plane, and thus to obtain the result we are looking for.
So we consider what happens as $R \to \infty$ and $\varepsilon \to 0$. We get that

$$2\int_{\varepsilon}^{R} \frac{1-\cos x}{x^2} dx \to 2\int_{0}^{\infty} \frac{1-\cos x}{x^2} dx$$

Term 3

$$\int_{\gamma_R^+} f(z) dz = \int_{\gamma_R^+} \frac{1-e^{iz}}{z^2} dz$$

A common trick is to consider the modulus of the integral if we seems incapable of saying anything about the integral itself.
So consider

$$
\begin{aligned}
\left|\int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}dz\right| &\leq \left(\max_{|z|=R}\left|\frac{1-e^{iz}}{z^2}\right|\right).\pi R \text{ where } \pi R \text{ is the perimeter of the semi-circle} \\
&= \pi R \max_{|z|=R} \frac{|1-e^{iz}|}{R^2} \text{ ,as } |z^2| = R^2 \\
&= \frac{\pi}{R}\max_{|z|=R}|1-e^{iz}| \ (*)
\end{aligned}
$$

Now we consider just $|1-e^{iz}|$ and try to deduce something about it.

$$
\begin{aligned}
|1-e^{iz}| = |1-e^{i(x+iy)}| &= |1-e^{ix}e^{-y}| \\
&\leq 1+|e^{ix}e^{-y}| = 1|e^{ix}||e^{-y}| \\
&= 1+e^{-y} \text{ since } |e^{ix}| = 1
\end{aligned}
$$

Now, we know that $y \geq 0$ as we are integrating over a contour located in the upper half-plane. Thus, $e^{-y} \leq 1$ and so $1+e^{-y} \leq 1+1 = 2$.
Then (*) becomes

$$
\left|.\int_{\gamma_R} \frac{1-e^{iz}}{z^2}dz\right| \leq \frac{\pi}{R}.2 = \frac{2\pi}{R}
$$

And as $R \to \infty$, we have that $\frac{2\pi}{R} \to 0$ and hence

$$
\int_{\gamma_R} f(z)dz \to 0
$$

So there is no contribution from this integral.
Term 4

$$
\int_{\gamma_\varepsilon} f(z)dz
$$

First of all, we have that γ_ε is the curve traversed clockwise. So we let C_ε be the same curve traversed anticlockwise and we have that this curve can be parametrised by

$$
z(t) = \varepsilon e^{it} \text{ for } 0 \leq t \leq \pi
$$

So we have

$$
\int_{\gamma_\varepsilon} f(z)dz = -\int_{C_\varepsilon} f(z)dz = -\int_{C_\varepsilon} \frac{1-e^{iz}}{z^2}dz
$$

Now again we want to consider what happens as $\varepsilon \to 0$. However, here parametrising the curve won't help as it will complicate the problem. It's not obvious how to proceed but here is a nice trick: We use the expansion of e^{iz}. We have that

$$
\begin{aligned}
-\int_{C_\varepsilon} \frac{1-e^{iz}}{z^2}dz &= -\int_{C_\varepsilon} \frac{1-\left(1+iz+\frac{(iz)^2}{2!}+\frac{(iz)^3}{3!}+\dots\right)}{z^2}dz \\
&= -\int_{C_\varepsilon} -\frac{i}{z} - \left(\frac{i^2}{2!}+\frac{i^3 z}{3!}+\frac{i^4 z^2}{4!}+\dots\right)dz \\
&= i\int_{C_\varepsilon} \frac{1}{z}dz + \int_{C_\varepsilon} E(z)dz \text{ where } E(z) = \frac{i^2}{2!}+\frac{i^3 z}{3!}+\frac{i^4 z^2}{4!}+\dots \\
&= i\int_0^\pi \frac{\varepsilon i e^{it}}{\varepsilon e^{it}} + \int_{C_\varepsilon} E(z)dz
\end{aligned}
$$

where we converted the line integral using the parametrisation from earlier.

$$
\begin{aligned}
&= i\int_0^\pi i dz + \int_{C_\varepsilon} E(z)dz \\
&= i\left[iz\right]_0^\pi + \int_{C_\varepsilon} E(z)dz \\
&= -\pi + \int_{C_\varepsilon} E(z)dz
\end{aligned}
$$

At this point we notice that we can actually solve the problem if we can show that $\int_{C_\varepsilon} E(z)dz \to 0$ as $\varepsilon \to 0$.
Again we are going to consider the modulus of the integral.

$$\left|\int_{C_\varepsilon} E(z)dz\right| \le \pi\varepsilon \max_{|z|=\varepsilon} |E(z)|$$

What can we say about $|E(z)|$?

$$E(z) = \frac{i}{2!}+\frac{i^3 z}{3!}+\frac{i^4 z^2}{4!}+\dots = \sum_{r=0}^{\infty} \frac{i^r}{r!} z^{r-2}$$

We can apply the ratio test

$$\left|\frac{i^{r+1}z^{r-1}}{(r+1)!}.\frac{r!}{i^r z^{r-2}}\right| = \left|\frac{iz^{r-1}}{(r+1)z^{r-2}}\right| = \frac{1}{(r+1)}|z| \to 0 \text{ as } r \to \infty$$

So we get that this series has an infinite radius of convergence and therefore $E(z)$ is continuous on $D(0,\varepsilon) \implies E(z)$ is bounded on $D(0,\varepsilon)$, say $|E(z)| \le M$ for some constant M. We now have that

$$\left|\int_{C_\varepsilon} E(z)dz\right| \le \pi\varepsilon M \to 0 \text{ as } \varepsilon \to 0$$

So the total contribution as $\varepsilon \to 0$ of

$$\int_{C_\varepsilon} f(z)dz$$

is 0.

Step 5: Combining all together, we get that

$$\begin{aligned}
0 &= 2\int_0^\infty \frac{1-\cos(x)}{x^2}dx + 0 + 0 + (-\pi) \\
&\implies 2\int_0^\infty \frac{1-\cos(x)}{x^2}dx = \pi \\
&\implies \int_0^\infty \frac{1-\cos(x)}{x^2}dx = \frac{\pi}{2}
\end{aligned}$$

6.11 (T)Cauchy's Inequalities

Theorem 20 (Cauchy's Inequlities)

$$\left|f^{(n)}(z_0)\right| \leq \frac{\left[n! \max_{|z-z_0|=R} |f(z)|\right]}{R^n}$$

Remark:
$max_{|z-z_0|=R}|f(z)|$ exists. This is the same as $\max_{t\in[0,2\pi]} |f(z_0 + Re^{it})|$, since $|f(z_0) + Re^{it}|$ is a continuous function, $f : [0, 2\pi] \to \mathbb{R}$, so it achieves a maximum (by Analysis 1)

Proof. First of all, by Cauchy's Integral Formula, we have that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

We want to end up with and inequality for the modulus of $f^{(n)}(z_0)$, we stick in the modulus signs:

$$\begin{aligned}
\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\int\limits_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \\
&= \frac{n!}{2\pi}\left|\int\limits_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \\
&\leq \frac{n!}{2\pi}.(2\pi R).\max_{|z-z_0|=R}\frac{|f(z)|}{|(z-z_0)^{n+1}|} \\
&= n!R.\max_{|z-z_0|=R}\frac{|f(z)|}{|R|^{n+1}} \\
&= \frac{n!}{R^n}\max_{|z-z_0|=R}|f(z)|
\end{aligned}$$

□

Definition 52 (Entire)
If f is holomorphic on the whole complex plane, then f is called entire.

6.12 (T)Liouville's Theorem and The Fundamental Theorem of Algebra

Theorem 21 (Liouville's Theorem)
If f is entire and bounded, then f = constant.

Proof. To find that f is constant we are going to prove that $f'(z) = 0$.
Let's use Cauchy's inequality on an integral about the circle with radius R, centered at z_0. We have

$$|f'(z)| \leq \max_{|z-z_0|=R}\frac{|f(z)|}{R}$$

Now, since f is bounded on $\mathbb{C}$, we can find a constant $M > 0$ such that

$$|f'(z_0)| \leq \frac{M}{R}$$

And now we note that, as $R \to \infty$, this goes to 0, and, hence

$$f'(z_0) = 0$$

Now, we took an arbitrary circle to begin with and a point z_0 to be its centre. But as we let $R \to \infty$, we are essentially covering the whole complex plane, and so we can take any z_0, i.e. $f'(z) = 0$, $\forall z \in \mathbb{C}$. The result follows. □

Example 51 (An application of Cauchy's Inequality and Liouville's Theorem)
Let f be entire and suppose we can find M, R_0, s.t.

$$|f(z)| \leq M|z|^{1/2}\ ,\ \forall z,\ \text{s.t. } |z| > R_0$$

i.e. we can bound the function outside the disk of radius R_0

Claim: $f(z)$ is a constant everywhere.

Proof. Fix $z_0 \in \mathbb{C}$. To show that $f'(z_0) = 0$, consider Cauchy's Inequality:

$$|f'(z_0)| \leq \max_{|z-z_0|=R} \frac{|f(z)|}{R} = \max_{|z-z_0|=R} \frac{M|z|^{1/2}}{R}$$

Now we want to say something about $|z|$. Let's express it as $|z| = |z - z_0 + z_0|$ and the we can apply the triangle inequality to get

$$|z - z_0 + z_0| \leq |z - z_0| + |z_0|$$

Now, if z is on the circle of radius R, then $|z - z_0| = R$ and so

$$|z - z_0| + |z_0| = R + |z_0|$$

So for any z outside the disk,

$$|z - z_0| + |z_0| < R + |z_0|$$

So we get

$$|z| \leq R + |z_0| \iff |z|^{1/2} \leq (R + |z_0|)^{1/2}$$

So now we can go back and say

$$|f'(z_0)| \leq \max_{|z-z_0|=R} \frac{|f(z)|}{R} = \max_{|z-z_0|=R} \frac{M|z|^{1/2}}{R} \leq \frac{(R + |z_0|)^{1/2}}{R}$$

And this goes to 0 as $R \to \infty$. $\implies$ $f'(z_0) = 0$. Therefore, $f'(z) = 0$, $\forall z \in \mathbb{C}$. $\implies$ $f(z) = const$ ∎

Theorem 22 (The Fundamental Theorem of Algebra)
Every non-constant polynomial (with constant coefficients) has a root in $\mathbb{C}$.

Proof. (By contradiction)
Assume that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0$$

has no roots in $\mathbb{C}$. We are going to look at $f(z) = \frac{1}{p(z)}$ We will show that $f(z)$ is entire and bounded and conclude, by Liovelle's Theorem, that $f(z) = k$. This, in turn would imply that $p(z) = \frac{1}{k}$, so our polynomial is also constant. But this is a contradiction.

Step 1 (f is entire)

$$\frac{1}{p(z)} = f(z) \text{ is entire}$$

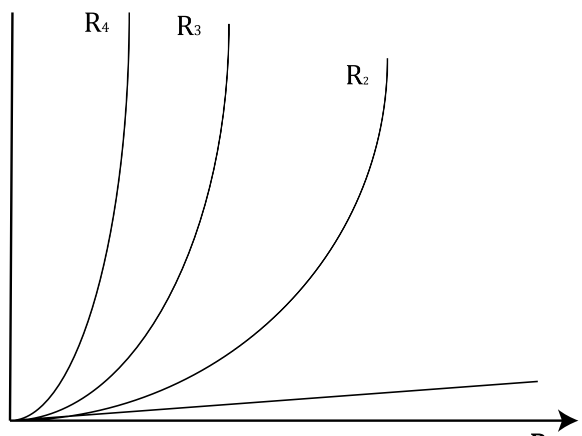
Now, the denominator is never zero, since our assumption is that $p(z)$ has no roots. $\implies$ f is entire.

Step 2 (f is bounded)
First of all, here is some intuition into the proof to follow:
Let $|z| = R$, where R is large (eventually going to ∞).

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

$$\begin{aligned}
&|a_n z^n| = |a_n|R^n \\
&|a_{n-1}z^{n-1}| = |a_{n-1}|R^{n-1} \\
&\vdots \\
&|a_0|
\end{aligned}$$

Since R is a large number, we expect that the first term above will be much bigger than the others. That is, the behaviour of the polynomial is pretty much determined by the leading term as we let $R \to \infty$.



End of intuition. Here is how we actually prove this:
Fix R and assume $|z| = R$.

$$\begin{aligned}
|p(z)| &= |a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \\
&\geq |a_n z^n| - |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \text{ by the triangle inequality}
\end{aligned}$$

Now,

$$\begin{aligned}
|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| &\leq |a_{n-1}||z^{n-1}| + |a_{n-2}||z^{n-2}| + \cdots + |a_1||z| + |a_0| \\
&= |a_{n-1}|R^{n-1} + |a_{n-2}|R^{n-2} + \cdots + |a_1|R + |a_0|
\end{aligned}$$

So plugging this back above, we get

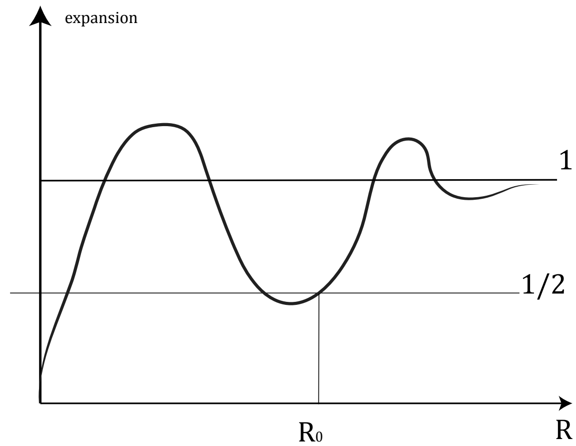
$$\begin{aligned}
|p(z)| &\geq |a_n|R^n - |a_{n-1}|R^{n-1} - |a_{n-2}|R^{n-2} - \cdots - |a_0| \\
&= |a_n|R^n\left(1 - \frac{|a_{n-1}|}{|a_n|}\frac{1}{R} - \frac{|a_{n-2}|}{|a_n|}\frac{1}{R^2} - \cdots - \frac{|a_0|}{|a_n|}\frac{1}{R^n}\right)
\end{aligned}$$

Now

$$\lim_{R\to\infty}\left(-\frac{|a_{n-1}}{|a_n|}\frac{1}{R} - \frac{|a_{n-2}|}{|a_n|}\frac{1}{R^2} - \cdots - \frac{|a_0|}{|a_n|}\frac{1}{R^n}\right) = 0$$

The Principle of Inertia says that $\exists R_0 > 0$ such that $\forall R$, $R > R_0$,

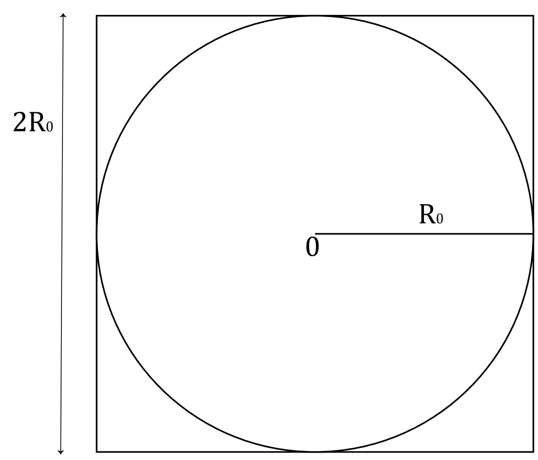
$$\left(1 - \frac{|a_{n-1}|}{|a_n|}\frac{1}{R} - \frac{|a_{n-2}|}{|a_n|}\frac{1}{R^2} - \cdots - \frac{|a_0|}{|a_n|}\frac{1}{R^n}\right) > \frac{1}{2}$$



So for $R > R_0$,

$$|p(z)| \geq \frac{1}{2}|a_n|R^n \iff \left|\frac{1}{p(z)}\right| \leq \frac{2}{|a_n|R^n} \leq \frac{2}{|a_n|R_0^n}$$

With this, we have proved that f is bounded for $R > R_0$, so now we only need to show that it is also bounded in the disk or radius R_0, i.e. in $D(0, R_0)$. Now, this disk is contained in the square with sides $2R_0$, i.e. $D(0, R_0) \subset [R_0, R_0] . [R_0, R_0] \left(= [R, R_0]^2\right)$
But on this square, $|f|$ is bounded. This is analogous to "continuous functions on closed intervals are bounded", from Analysis 1. This is so because $\frac{1}{p(z)}$ is continuous on the box $[R_0, R_0]^2$, and therefore it is bounded on it.



So, $\frac{1}{p(z)}$ is bounded on $D(0,R_0)$ by, say, k. So

$$\left|\frac{1}{p(z)}\right| \le k, \forall z \in D(0,R_0)$$

So now he have inequalities for f both inside and outside he disk of radius R_0, i.e. we have 2 inequalities covering essentially the whole complex plane. Let's turn them into a single inequality by taking $M = \max\left(k, \frac{2}{|a_n|R_0^n}\right)$.
Then $f \le M$ on the whole complex plane. So we have proved that f is bounded.
Hence, we have now proved that f is both entire and bounded, which implies (Liouville's Theorem) that f =constant and so $p(z)$ = constant. This is a contradiction, since the statement of the the theorem requires $p(z)$ to be nonconstant. □

Corollary 10
Every polynomial over $\mathbb{C}$ of degree $n \ge 1$ can be factorised as

$$p(z) = a_n(z-w_1)(z-w_2)\cdots(z-w_n)$$

where $w_1,\dots,w_n$ are the roots of the polynomial and there are exactly n of them.

Proof. By the Fundamental Theorem of Algebra, we can find a root w_1 for the polynomial. i.e. we can write

$p(z) = (z-w_1)Q(z)+R(z)$, where $Q(z)$ is the quotient and $R(z)$ is the remainder of the division

Since w_1 is a root, we have $p(w_1) = 0$, so

$$0 = p(w_1) = 0.Q(w_1) + R \implies R = 0$$

So

$$p(z) = (z-w_1)Q(z)$$

Now $deg(Q(z)) = n-1$ and we can write

$$p(z) = (z-w_1)(z-w_2).Q^1(z), \text{ where } deg(Q^1(z)) = n-2$$

Continue in this way until

$$p(z) = (z-w_1)(z-w_2)\dots(z-w_n).C \text{ , where C is a constant quotient}$$

Now compare

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0 = C(z-w_1)\dots(z-w_{n-1})(z-w_n)$$

So z^n has coefficient a_n on the LHS, while it has a coefficient C on the RHS. Therefore, $a_n = C$. □

6.13 Power Series Expansion of Holomorphic Functions

Theorem 23
Let f be holomorphic on an open set containing the closed disk $D(z_0, R)$. Then f has a power series expansion at z_0.
$\forall z \in D(z_0, R)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

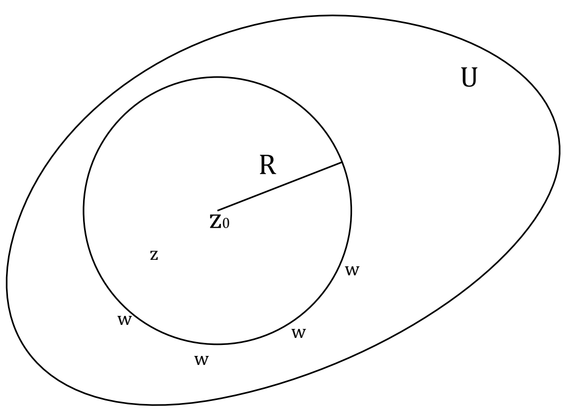
Proof. By Cauchy's Integral Formula

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(w)}{w - z} dw \\ &= \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(w)}{w - z_0 - (z - z_0)} dw \\ &= \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(w)}{w - z_0} . \frac{1}{1 - \frac{z - z_0}{w - z_0}} \end{aligned}$$

Now we notice that the term $\frac{1}{z - \frac{z-z_0}{w-z_0}}$ is the sum of the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n$$

for $r = \frac{z-z_0}{w-z_0}$, if $|r| < 1$.
Now, looking at our picture: The w's lie on the circle, while z is inside the circle, and so is z_0, which is the centre. So we have that $|z - z_0| < |w - z_0|$ and so r is indeed less than 1 and the sum we wrote above is valid.



Therefore, now we have

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_R(z_0)} \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi i}\int_{C_R(z_0)} \frac{f(w)}{w-z_0}.\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^n} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_R(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\right)dw(z-z_0)^n
\end{aligned}
$$

which is what we wanted to prove. However, we still need to prove that the last step we did was legitimate. We need to show that

$$\sum_{n=0}^{N}\int_{C_R(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw.(z-z_0)^n \to \int_{C_R(z_0)} \sum_{n=0}^{\infty}\frac{f(w)}{(w-z_0)^{n+1}}(z-z_0)^n dw \text{ as } N\to\infty$$

Consider the modulus of the difference

$$
\begin{aligned}
&\left|\sum_{n=0}^{N}\int_{C_R(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}(z-z_0)^n dw - \int_{C_R(z_0)} \sum_{n=0}^{\infty}\frac{f(w)}{(w-z_0)^{n+1}}(z-z_0)^n dw\right| \\
&= \left|.\int_{C_R(z_0)}\left(\sum_{n=0}^{\infty}-\sum_{n=0}^{N}\right)\frac{f(w)}{(w-z_0)^{n+1}}(z-z_0)^n dw\right| \\
&= \left|.\int_{C_R(z_0)}\sum_{N+1}^{\infty}\frac{f(w)}{(w-z_0)^{n+1}}(z-z_0)^n dw\right| \\
&\le 2\pi R.\max_{|w-z_0|=R}\left|\sum_{n>N}\frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}}\right| \\
&\le 2\pi R\max_{|w-z_0|=R}\sum_{n>N}\left|\frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}}\right| \\
&\le 2\pi R\max_{|w-z_0|=R}\frac{|f(w)|}{|w-z_0|}.\sum_{n>N}\left|\frac{z-z_0}{w-z_0}\right|^n \\
&= 2\pi R\max_{|w-z_0|=R}\frac{|f(w)|}{R}.\sum_{n>N}\left|\frac{z-z_0}{w-z_0}\right|^n \\
&= 2\pi R\max_{|w-z_0|=R}\frac{|f(w)|}{R}.\sum_{n>N}|r|^n \\
&= 2\pi\max_{|w-z_0|=R}|f(w)|\sum_{n>N}|r|^n
\end{aligned}
$$

where

$$\sum_{n>N}|r|^n = \frac{|r|^{N+1}}{1-|r|}$$

so

$$|r| = \left|\frac{z - z_0}{w - z_0}\right| < 1$$

$$\lim_{N\to\infty} |r|^{N+1} = 0$$

and so

$$\lim_{N\to\infty} \frac{|r|^{N+1}}{1 - |r|} = 0$$

Thus we have shown that

$$\sum_{n=0}^{N} \int \to \int \sum_{n=0}^{\infty}$$

But also,

$$\sum_{n=0}^{N} \int \to \sum_{n=0}^{\infty} \int$$

And these two imply that

$$\sum_{n=0}^{\infty} \int = \int \sum_{n=0}^{\infty}$$

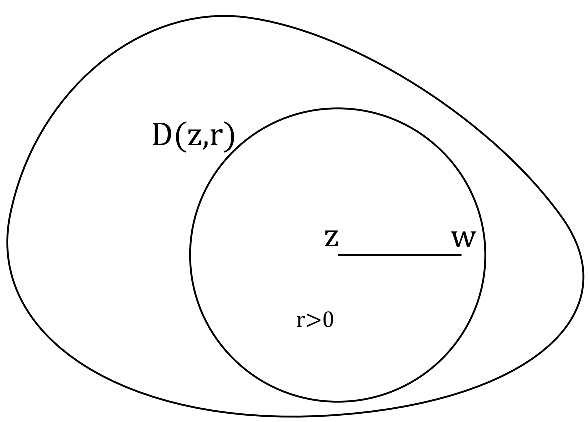
□

Theorem 24 (Morera's Theorem)
Let f be continuous on a region Ω and

$$\int_{\gamma} f(z)dz = 0$$

for all γ, closed contours in Ω, with interior in Ω.
Then f is holomorphic.

Proof. Let $D(z, r)$ be a disk (a convex set) inside Ω. For f continuous on a convex set, we can construct a primitive function F on $D(z, r)$ such that $F' = f$.



$$F(w) = \int_{[z,w]} f(u)du$$

So F in holomorphic and by the previous theorem, F has a power series expansion. Therefore, it is infinitely many times differentiable. But $f = F'$, so f is also infinitely many times differentiable, i.e. it is holomorphic. $\square$

Discussion

If f is holomorphic on $D(z_0, r)$, $r > 0$, we can express f as a power series centred at z_0 as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$. Then we have 2 cases:
Case 1: If $a_n = 0$, $\forall n$, then $f(z) = 0$, $\forall z \in D(z_0, r)$
Case 2: If $f(z) \neq 0$ $\forall z \in D(z_0, r)$, then $\exists a_n$ which are not 0.
Let N be the smallest such n for which $a_N \neq 0$, while $a_0 = a_1 = \cdots = a_{N-1} = 0$. Then

$$\begin{aligned} f(z) &= a_N (z - z_0)^N + a_{N+1}(z - z_0)^{N+1} + \dots \\ &= (z - z_0)^N \left(a_N + a_{N+1}(z - z_0) + a_{N+2}(z - z_0)^2 + \dots \right) \\ &= (z - z_0)^N g(z) \end{aligned}$$

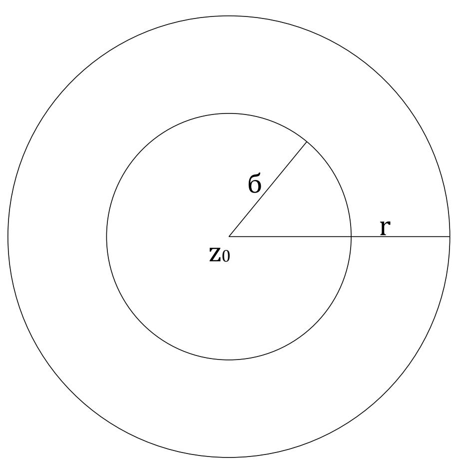
Here $g(z)$ has the same coefficients as the series of $f(z)$, so $g(z)$ has the same radius of convergence $r > 0$, so $g(z)$ is holomorphic on $D(z_0, r) \implies g$ is continuous on $D(z_0, r)$.

$$\lim_{z \to z_0} g(z) = g(z_0) = a_N \neq 0$$

By the Inertia Principle. $\exists \delta > 0$ s.t. $g(z) \neq 0$. for $z \in D(z_0, \delta)$.
So

$$f(z) = (z - z_0)^N g(z)$$

with $g(z)$ holomorphic and non-vanishing on $D(z_0, \delta)$.



Note: See page 39 and complete with more explanations. Not everything from lectures written down.

Theorem 25
If f is holomorphic on $D(z_0, r)$, $r > 0$ and f doesn't vanish identically (i.e. it is not the case that $f(z) = 0, \forall z \in D(z_0, r)$), then $\exists \delta > 0$, and a non-vanishing holomorphic function $g(z)$ on $D(z_0, \delta)$ and also a unique $N \geq 0$ with

$$f(z) = (z - z_0)^N g(z) \ , \ \forall z \in D(z_0, \delta)$$

Proof. We have already given most of the proof in the previous discussion. The only thing that remains to be proved in uniqueness.
Assume

$$f(z) = (z - z_0)^N g(z)$$

and

$$f(z) = (z - z_0)^k h(z)$$

with g and h holomorphic and non-vanishing on some $D(z_0, \delta)$, $\delta > 0$. Then $N = k$. Why is this true?
Suppose that $N > k$. Then we can cancel out $(z - z_0)^k$ from

$$f(z) = (z - z_0)^N g(z) = (z - z_0)^k h(z)$$

to get

$$(z - z_0)^{N-k} g(z) = h(z) \text{ ,where } N - k > 0$$

Then, taking the limits of both sides, we get that

$$\lim_{z \to z_0} (z - z_0)^{N-k} g(z) = \lim_{z \to z_0} h(z)$$

This implies that

$$0.g(z_0) = h(z_0) \implies h(z_0) = 0$$

but we were assuming that h is non-vanishing. This is a contradiction.
The case $k > N$ gives a similar contradiction.
Hence, $k = N$ □

6.14 Singularities and Zeroes

Definition 53
If $N \geq 1$, we say that $f(z)$ has a zero at z_0 of order N.
If $N = 1$, we say that $f(z)$ has a simple zero at z_0

Definition 54 (Removable Singularity)
If $f(z)$ is defined and holomorphic on the punctured disk $D'(z_0, r) = \{z; 0 < |z - z_0| < r\}$ and one can define $f(z_0)$ such that f is holomorphic on the whole disk $D(z_0, r)$, then z_0 is called a removable singularity.

Theorem 26
Suppose that $f(z)$ is analytic in the region Ω' obtained by omitting a point a from the region Ω. A necessary and sufficient condition that there exists an analytic function in Ω which coincides with $f(z)$ in Ω' is that

$$\lim_{z \to a} (z - a) f(z) = 0$$

The extended function is uniquely determined.
(Theorem form Ahlros, p 124)

This theorem is useful for actually checking whether something is a removable singularity or not. In class we mentioned that if $\lim_{z \to a} f(z) = \infty$, where a is the singularity means that the singularity is not removable. However, the theorem above provides a rigorous check for removable singularities. Note that as long as $\lim_{z \to a} f(z)$ is computable, and not ∞, we can safely (but non-rigorously) conclude that the singularity is removable. Note also that, by computing the above limit, if it exists, we can find the value of $g(a)$, i.e. the value we want to assign to the function at a removable singularity (where g is our "extended" f, as in the definition above.)

The following are a few examples done first in the way we proved them in class (Step 1) and then applying the theorem above (Step 2), and then we find the value of $g(a)$ (Step 3) in Example 1 only.

Example 52

$$f : \mathbb{C} - \{0\} \to \mathbb{C}, f(z) = z$$

Part 1:
Define $g : \mathbb{C} \to \mathbb{C}$ s.t. $\forall z \in \mathbb{C} - \{0\}$, $g(z) = f(z)$ and $g(z) = 0$ for $z = 0$
Then $g(z) = z$, $\forall z \in \mathbb{C}$, and hence, g is entire. So we have a removable singularity at 0.
Part 2:
Consider

$$\lim_{z \to 0} (z - 0) z = \lim_{z \to 0} z^2 = 0$$

So 0 is indeed a removable singularity.
Part 3:
To find the actual value we want to take for $g(0)$, we compute:

$$\lim_{z \to 0} z = 0$$

So we take $g(0) = 0$.
Note that in this case it was obvious what value we wanted to take for $g(0)$ but in some more complicated cases, computing the limit is actually required.

Example 53

$$f : \mathbb{C} - \{0\} \to \mathbb{C}, f(z) = \frac{1}{z}$$

Part 1: The point where we have a singularities is 0.

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{1}{z} = \infty$$

(note: not formal proof of infinite limit. See definition below) So we don't have a removable singularity at 0.
Part 2:

$$\lim_{z \to 0} (z - 0) \frac{1}{z} = \lim_{z \to 0} (1) = 1 \neq 0$$

So we don't have a removable singularity at 0.
Part 3:
This is the same as part 1, so in class we used this as proof that the singularity is not removable.

Example 54

$$f(z) = \frac{z + 1}{z^3.(z^2 + 1)}$$

The singularities are at 0, i, $-i$.
Part 1:
At 0:

$$\begin{aligned} \lim_{z \to 0} f(z) &= \lim_{z \to 0} \frac{z + 1}{z^3(z^2 + 1)} \\ &= \left(\lim_{z \to 0} \frac{1}{z^3}\right) . \left(\lim_{z \to 0} \frac{z + 1}{z^2 + 1}\right) \\ &= \infty.(1) = \infty \end{aligned}$$

We don't get a finite value for the limit, so there is no removable singularity.
At i:

$$\begin{aligned} \lim_{z \to i} f(z) &= \lim_{z \to i} \frac{z + 1}{z^3(z^2 + 1)} \\ &= \lim_{z \to i} \frac{z + 1}{z^3} \frac{1}{z + i} \frac{1}{z - i} \\ &= \left(\lim_{z \to i} \frac{z + 1}{z^3}\right) . \left(\lim_{z \to i} \frac{1}{z + i}\right) . \left(\lim_{z \to i} \frac{1}{z - i}\right) \\ &= \infty.\infty.\infty \\ &= \infty \end{aligned}$$

So again we don't have a removable singularity.
Exactly similar calculation shows that for $z = -i$, we also get

$$\lim_{z \to -i} f(z) = \infty$$

And so we don't have any removable singularities in this example.

Part 2:
Let's apply the theorem to the 3 points in turn:

$$\lim_{z \to 0}(z-0)f(z) = \lim_{z \to 0} \frac{z+1}{z(z^2+1)} = \infty \neq 0$$

$$\lim_{z \to i}(z-i)f(z) = \lim_{z \to i} \frac{z+1}{z^3(z+i)} = \infty \neq 0$$

$$\lim_{z \to -i}(z+i)f(z) = \lim_{z \to -i} \frac{z+1}{z^3(z-i)} = \infty \neq 0$$

$\implies$ that the points are not removable singularities.
(note: not formal proofs of infinite limit. See definition below)

Example 55
An additional example

$$f(z) = \frac{\sin z}{z}$$

Applying the Theorem to check if we have a singularity at $z = 0$:

$$\begin{aligned} \lim_{z \to 0}(z-0)f(z) \qquad & = \lim_{z \to 0} z.\frac{\sin z}{z} \\ = \lim_{z \to 0} \sin z \qquad & = \lim_{z \to 0}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right) \\ = 0 \end{aligned}$$

Hence, 0 is a removable singularity. To check the value of $g(0)$, we compute the limit:

$$\begin{aligned} \lim_{z \to 0} f(z) &= \lim_{z \to 0} \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots}{z} \\ &= \lim_{z \to 0} 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \\ &= 1 \end{aligned}$$

So we take $g(0) = 1$. And thus we define

$$g : \mathbb{C} \to \mathbb{C} = f : \mathbb{C} - \{0\} \to \mathbb{C}, \forall z \in \mathbb{C} - \{0\}$$

and $g(z) = 1$ for $z = 0$, so g is holomorphic everywhere.

Definition 55
We define

$$\lim_{z \to z_0} f(z) = \infty$$

to mean

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } 0 < |z - z_0| < \delta \implies |f(z)| > M$$

Definition 56 (Isolated singularity)
If f is holomorphic on the punctured disk $D'(z_0, r) = \{z, 0 < |z - z_0| < r\}$, then z_0 is called an isolated singularity.

> **Additional 3**
> Easy way of thinking about this: If we have a function with more than 1 singularity, we want those to be relatively further apart to be consider isolated singularities (not how the word "isolated" makes sense). This means that around any of the singularities, we can draw a punctured disk with an arbitrarily small radius such that the disk doesn't contain any other singularities.

Example 56

$$f(z) = Log(z) = Log|z| + iArg(z), -\pi < Arg(z) \leq \pi$$

There is a singularity at $z_0 = 0$, but this is not an isolated singularity as no matter how small an open disk we put around 0, f still won't be holomorphic on it since Log requires a cut from $-\infty$ to 0, which goes through any circle centred at the origin, and so f is not holomorphic there.

Example 57

$$f(z) = \frac{1}{\sin(\frac{\pi}{z})}$$

First let's find all the singularities:

$$\sin(\frac{\pi}{z}) = 0 \iff \frac{\pi}{z} = \arcsin(0) \iff n\pi = \frac{\pi}{z} \text{ for } n \in \mathbb{N}$$

So all the points $z = \frac{1}{n}$ for $n \in \mathbb{N}$ are singularities.
There is also a singularity at $z = 0$

Claim: Only $z = \frac{1}{n}$ are an isolated singularities.
Consider the following argument: The numbers $z = \frac{1}{n}$ all lie on the real axis and are all between -1 and 1. Since $n \in \mathbb{N}$, we can't cover the whole line segment $[-1, 1]$ with numbers $z = \frac{1}{n}$, since there are also irrational numbers in this interval. But this means that around any of the numbers $\frac{1}{n}$ we can draw some circle such that the interior of the circle doesn't include any other singularities but the one at the centre. And since we're interested in f being holomorphic on a punctured disk, we can just take the disk to be, centred at a given $\frac{1}{n}$, with radius extending to the singularity to the left and to the right of the one we are considering. But by the definition of isolated singularity, this means that all of the points $z = \frac{1}{n}$ are isolated singularities.
On the other hand, consider the singularity at $z = 0$. If we draw any disk around $z = 0$, this disk will contain some other singularities (at least 2) of the form $z = \frac{1}{n}$. So f can't be made holomorphic on any punctured disk with 0 at its centre. So 0 is not an isolated singularity.

Definition 57 (Pole)
We say that f has a pole at z_0 if f is defined at some punctured disk $D'(z_0, r) \to \mathbb{C}$ and

$$\frac{1}{f} : D'(z_0, r) \to \mathbb{C}$$

can be defined at z_0 by

$$\frac{1}{f}(z_0) = 0$$

and is holomorphic on $D(z_0, \delta)$ for $\delta > 0$

> **Additional 4**
> A **meromorphic** function on an open subset D of the complex plane is a function that is holomorphic on all D except a set of isolated singularities, which are poles for the function.
>
> Note that removable singularities are just a type of isolated singularities. Here is the full classification:
>
> **Definition 58** (Isolated singularity)
> Let U be an open subset of a Riemann surface and let $z_0 \in U$.
> A holomorphic function $f : U - \{z_0\} \to \mathbb{C}$ is said to have an isolated singularity at z_0.
>
> **Types of isolated singularities:**
>
> - Removable singularity
> - A pole:
>   If $\lim_{z \to z_0} f(z) = \infty$, then the point z_0 is said to be a pole of $f(z)$ and we set

$f(a) = \infty$.

- An essential singularity:
  This is an isolated singularity which is neither removable nor a pole.

An important point is that a "pole" is actually the same thing as a removable singularity, if we think of our function as a map that takes values on the Riemann sphere (which is the complex plane with a point at ∞ added; the complex structure near ∞ comes from the map $z \to \frac{1}{z}$).

So a function that has a removable singularity or a pole at z_0 doesn't have a "real" singularity there at all; rather, we can extend the function to an analytic or meromorphic function in z_0. If we cannot extend the function in this way, the singularity is indeed "essential"; i.e., we cannot get rid of it.
(End of additional)

Theorem 27
If f has a pole at z_0, then $\exists$ a small $\delta > 0$ and a holomorphic function $h(z)$ on $D(z_0, \delta)$ which doesn't vanish of $D(z_0, \delta)$ and there exists a unique $N \in \mathbb{N}$ with

$$f(z) = (z - z_0)^{-N} h(z)$$

where $z \in D(z_0, \delta)$

Proof. $\frac{1}{f}$ can be written as $(z - z_0)^{-N} g(z)$ with $N \geq 1$, by the previous Theorem.
Here $g(z)$ is holomorphic and it doesn't vanish on the disk $D(z_0, \delta)$

$$f(z) = (z - z_0)^{-N} \frac{1}{g(z)}$$

Set $h(z) = \frac{1}{g(z)}$. As $g(z)$ is holomorphic and $g(z \neq 0)$ on the disk, $h(z)$ is also holomorphic and not equal to 0 on the disk. □

Definition 59 (Order of the pole)
The N which appears in the proof above is called the order of the pole.
If $N = 1$, then the pole is called a simple pole.

Additional 5
In the following examples we use the fact that we already found the isolated, non-removable singularities of the function and we are trying to show that they are in fact poles by trying to represent the function in the form $f(z) = (z - z_0)^{-N} \frac{1}{g(z)}$ as in the theorem above. (We never use the definition of the pole as being a limit.) By writing the function in this form, we can see what the order of the pole is.
Note that it seems like we are using the theorem, so we are assuming that we already know the poles and that they are the non-removable isolated singularities, and yet we end up proving that these points are the poles. This seems like a wrong approach. The theorem states that we can represent f as $f(z) = (z - z_0)^{-N} \frac{1}{g(z)}$ if we already know that the point z_0 is a removable singularity but we use it to

> prove that. Adopt this approach for the purposes of the course but check with the other definitions when possible.
> (End of additional)

Example 58

$$f(z) = \frac{1}{z}$$

We have seen that 0 is an isolated singularity and it's non-removable.
We can write

$$f(z) = (z-0)^{-1}.1$$

Where our $h(z) = 1$ is holomorphic and never 0. So 0 is a pole, and it is in fact a simple pole.

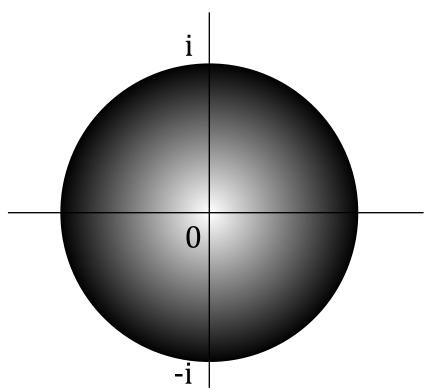
Example 59

$$f(z) = \frac{z+1}{z^3(z^2+1)}$$

We found that f has non-removable singularities at 0, i, $-i$. Let's look at each of them:
At 0:
We can represent f as

$$f(z) = (z-0)^{-3}.\frac{z+1}{z^2+1}$$

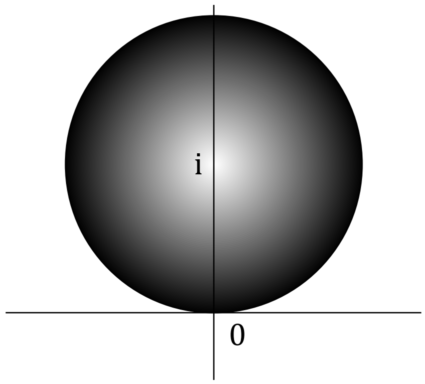
where our function $h(z) = \frac{z+1}{z^2+1}$. This function is holomorphic on $\mathbb{C} - \{+i, -i\}$. To fulfill the conditions of the theorem, we need to find a $D(0,\delta)$ s.t. $h(z)$ is holomorphic on it and doesn't vanish on it. How do we find such a disk? Well, the denominator of h vanishes at $i, -i$ so we can take the radius of the disk to be 1. i.e. take the disk $D(0,1)$.
So 0 is a pole of order 3.



At i:
We can represent f as

$$f(z) = (z-i)^{-1}.\frac{z+1}{z^3(z+i)}$$

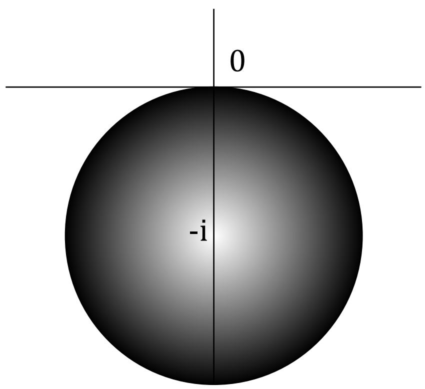
where the function $h(z) = \frac{z+1}{z^3(z+i)}$. This function is holomorphic on $\mathbb{C} - \{0, -i\}$. The disk we can put around i s.t. h is holomorphic and non-vanishing on it is $D(i, 1)$
So i is a simple pole.



At $-i$:
We can represent f as:

$$f(z) = (z+i)^{-1}.\frac{z+1}{z^3(z-i)}$$

Where the function $h(z) = \frac{z+1}{z^3(z-i)}$. This function is holomorpic on $\mathbb{C} - \{0, i\}$. The disk we can put around $-i$ s.t. h is holomorphic and non-vanishing on it is $D(-i, 1)$.



Theorem 28
If f has a pole of order N at z_0, then we can write

$$f(z) = \frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-N+1}}{(z-z_0)^{N-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z)$$

with a_{-N}, a_{-N+1}, $\ldots$, a_{-1} uniquely determined and $G(z)$ holomorphic in some $D(z_0, \delta)$ for $\delta > 0$

Proof. We have

$$f(z) = (z-z_0)^{-N}.h(z)$$

with $h(z)$ holomorphic and differentiable from 0 on $D(z_0, \delta)$.

$$h(z) = A_0 + A_1(z - z_0) + A_2(z - z_0)^2 + \dots$$

So,

$$f(z) = \frac{A_0}{(z - z_0)^N} + \frac{A_1}{(z - z_0)^{N-1}} + \frac{A_2}{(z - z_0)^{N-2}} + \dots + \frac{A_{N-1}}{(z - z_0)} + G(z)$$

Set
$a_{-N} = A_0$, $a_{-N+1} = A_1$, $a_{-N+2} = A_2$, $\dots$ $a_{-1} = A_{N-1}$ □

6.15 Residues and The Residue Theorem

Definition 60 (Residue)
The numerator of the term $\frac{a_{-1}}{z-z_0}$, a_{-1} is called the residue of f at the pole z_0. We write

$$res(f, z_0) = a_{-1}$$

Definition 61

$$\dots + \frac{a_{-1}}{z - z_0}$$

is called the principal part.

Theorem 29
If z_0 is a simple pole of $f(z)$, the residue is given by

$$res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

If z_0 is a pole of order N, then the residues is given by

$$res(f, z_0) = \frac{1}{(N-1)!} \lim_{z \to z_0} \frac{d^{(N-1)}}{dz^{(N-1)}} (z - z_0)^N f(z)$$

Proof. Part 1: $N = 1$
Since f has a simple pole,

$$f(z) = \frac{a_{-1}}{z - z_0} + G(z)$$

And multiplying through by the denominator, we get

$$(z - z_0) f(z) = a_{-1} + G(z)(z - z_0)$$

And now we take the limits of both sides as $z \to z_0$. We get

$$\begin{aligned}
\lim_{z \to z_0} [(z - z_0) f(z)] &= \lim_{z \to z_0} [a_{-1} + G(z).(z - z_0)] \\
&= a_{-1} + \lim_{z \to z_0} (z - z_0) G(z) \\
&= a_{-1} + G(z_0) \lim_{z \to z_0} (z - z_0) \text{ , as } G(z) \text{ is continuous} \\
&= a_{-1} + 0 \text{ , since } \lim_{z - z_0} (z - z_0) = 0
\end{aligned}$$

Thus,

$$a_{-1} = \lim_{z \to z_0} [(z - z_0) f(z)]$$

Part 2: $N > 1$

$$(z - z_0)^N f(z) = a_{-N} + a_{-N+1}(z - z_0) + a_{-N+2}(z - z_0)^2 + \ldots + a_{-1}(z - z_0)^{N-1} + (z - z_0)^N G(z)$$

Differentiating once, we get

$$\frac{d\left[(z - z_0)^N f(z)\right]}{dz} = a_{-N+1} + 2a_{-N+2}(z-z_0) + \ldots + (N-1)a_{-1}(z-z_0)^{N-2} + N(z-z_0)^{N-1}G(z) + (z-z_0)^N . G'(z)$$

Differentiating again, we get

$$\frac{d^2\left[(z - z_0)^N f(z)\right]}{dz^2} = 2a_{-N+2} + 6a_{-N+3}(z - z_0)$$
$$+ \ldots + (N-2)(N-1)a_{-1}(z - z_0)^{N-3} + N(N-1)(z - z_0)^{N-2}G(z) + 2N(z - z_0)^{N-1}G'(z) + (z - z_0)^N G''(z)$$

We continue in this way until the $N - 1^{\text{th}}$ derivative:

$$\frac{d^{N-1}\left[(z - z_0)^N f(z)\right]}{dz^{N-1}} = (N-1)(N-2)(N-3)\ldots 1 a_{-1} + \text{ terms containing at least one power of } (z - z_0),$$
which is multiplied either by G or a higher derivative of G

and all of these other terms go to 0 as $z \to z_0$. So we have

$$\lim_{z \to z_0} \left[\frac{d^{N-1}(z - z_0)^N f(z)}{dz^{N-1}}\right] = (N-1)! a_{-1}$$

Thus

$$a_{-1} = \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^N} (z - z_0)^N f(z)$$

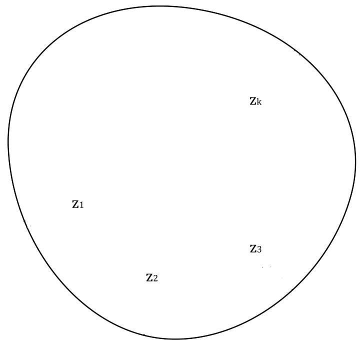
□

Theorem 30 (The Residue Theorem)
Let f be holomorphic on an open set containing a toy contour γ and its interior except for the poles $z_1, \ldots, z_k$ inside γ. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{i=1}^{k} res(f, z_i)$$

This is called The Residue Formula.

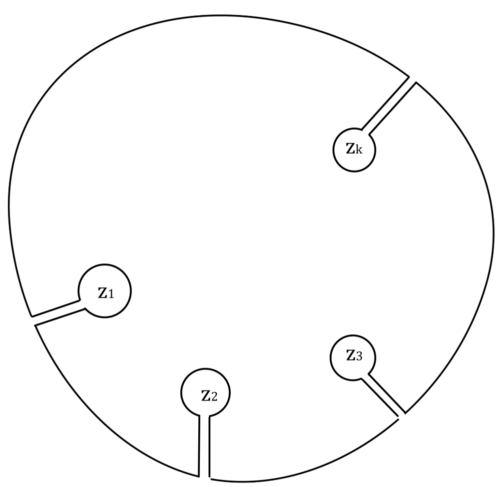
Proof. **Step 1**
First of all, pick at region Ω with contour γ along which to integrate. Let the points z_1, $z_2, \ldots, z_k$ be the poles of the function we want to integrate.



$f(z)$ is clearly not holomoprpic on this region as it is not even defined on the poles (they are not removable singularities). However, $f(z)$ is holomorpic on the keyhole region γ', since it is a region that excludes all singularities. So we have that

$$\int_{\gamma'} f(z)dz = 0$$

by Cauchy's Theorem.
We represent the keyhole region as follows:
We are traversing the whole contour anticlockwise, which means that all of the small almost-circles around the singularities are traversed clockwise.
Each small almost-circle is called $C_\varepsilon(z_j)$, where z_j is a pole, $1 \leq j \leq k$, and the pole is at the centre of the circle.



The width of the corridor going to the keyholes is very small, $\delta \to 0$.
When we let the width of the corridor tend to 0, i.e. $\delta \to 0$, we essentially make the two line segments which make up each corridor become the same line segment; but since the original two line segments were traversed in the opposite directions, the integrals along them actually cancel out. The contribution of the integrals along all the corridors is thus

0.
Then we have

$$\int_{\gamma} f(z)dz - \int_{C_\varepsilon(z_1)} f(z)dz - \int_{C_\varepsilon(z_2)} f(z)dz - \cdots - \int_{C_\varepsilon(z_k)} f(z)dz = \int_{\gamma'} f(z)dz = 0$$

Note that the − signs come from the fact that we are traversing the small circles clockwise.

Step 2

We want to show that for each j, $1 \le j \le k$, we have that

$$\int_{C_\varepsilon(z_j)} f(z)dz = 2\pi i res(f, z_j)$$

Now, we have

$$f(z) = \frac{a_{-n}}{(z - z_j)^N} + \ldots + \frac{a_{-1}}{z - z_j} + G(z)$$

where $G(z)$ is holomorpic in $D(z_j, \delta)$. Taking the integral on both sides, we get

$$\int_{C_\varepsilon(z_j)} f(z)dz = \int_{C_\varepsilon(z_j)} \frac{a_{-N}}{(z - z_j)^N} dz + \ldots + \int_{C_\varepsilon(z_j)} \frac{a_{-1}}{z - z_j} dz + \int_{C_\varepsilon(z_j)} G(z)dz$$

Now,

$$\int_{C_\varepsilon} G(z)dz = 0$$

since $G(z)$ is holomorphic on $D(z_j, \delta)$, by Cauchy's Theorem.
All the other integrals which have in the denominator a power of $(z - z_j)$ higher than 1 are also 0, since we can find an antiderivative for all of them. For example, the antiderivative of

$$\int_{C_\varepsilon(Z_j)} \frac{a_{-n}}{(z - z_j)^N} dz$$

is

$$F = \frac{a_{-N}}{-N + 1}(z - z_j)^{-N+1}$$

So the only non-zero term comes from

$$\int_{C_\varepsilon(z_j)} f(z)dz$$

and it is in fact equal to $2\pi i.a_{-1}$, by Cauchy's Integral Formula.
So we get

$$\int_{C_\varepsilon(z_j)} f(z)dz = 2\pi i a_{-1} = 2\pi i.res(f, z_j)$$

Step 3

When we go back to the expression we ended up with in Step 1, we get that

$$\begin{aligned}\int_{\gamma} f(z)dz &= \int_{C_\varepsilon(z_1)} f(z)dz + \int_{C_\varepsilon(z_2)} f(z)dz + \ldots + \int_{C_\varepsilon(z_k)} f(z)dz \\ &= 2\pi i.res(f, z_1) + 2\pi i.res(f, z_2) + \ldots + 2\pi i.res(f, z_k) \\ &= 2\pi i.\sum_{j=1}^{k} res(f, z_j)\end{aligned}$$

$\square$

Example 60

$$f(z) = \frac{1}{9 + z^2}$$

The isolated singularities are $3i$, $-3i$. Let's look at the them in turn, find if they are poles and of what order and find $h(z)$ and a disk D on which h is holomorpic and non-vanishing and calculate the residues.

What is the radius of convergence of the power series of f centred at 0? Centred at i?

Calculate

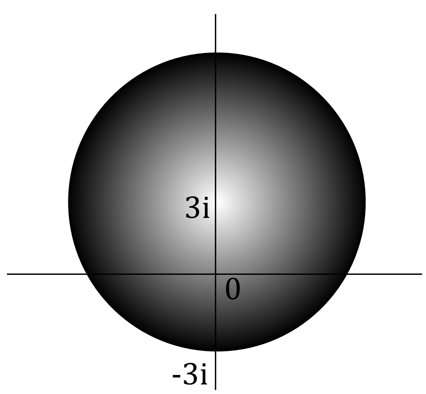
$$\int_{-\infty}^{\infty} \frac{1}{9 + x^2} dx$$

1. Singularity $3i$:

We can write f as

$$f(z) = (z - 3i)^{-1}.\frac{1}{z + 3i}$$

where $h(z) = \frac{1}{z+3i}$ and it holomorphic on the disk $D(3i, 6)$.



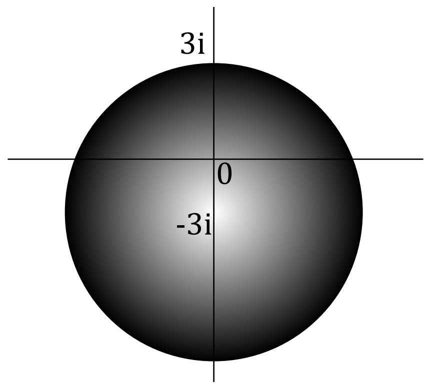
So we have a pole at $3i$ which is a simple pole. We can calculate the residue as

$$\begin{aligned}
res(f,3i) &= \lim_{z\to 3i}(z-3i)^{-1}.f(z)\\
&= \lim_{z\to 3i}\frac{1}{z-3i}\frac{1}{z-3i}\frac{1}{z+3i}\\
&= \lim_{z\to 3i}\frac{1}{z+3i}\\
&= \frac{1}{3i+3i}=\frac{1}{6i}
\end{aligned}$$

2. Singularity $-3i$:
We can write f as

$$f(z)=(z+3i)^{-1}.\frac{1}{z-3i}$$

where $h(z)=\frac{1}{z-3i}$ and it is holomorphic on the disk $D(-3i,6)$.



So we have a simple pole at $-3i$. We can calculate the residue as

$$\begin{aligned}
res(f,-3i) &= \lim_{z\to -3i}(z+3i)^{-1}.f(z)\\
&= \lim_{z\to -3i}\frac{1}{z+3i}\frac{1}{z-3i}\frac{1}{z+3i}\\
&= \lim_{z\to -3i}\frac{1}{z-3i}\\
&= \frac{1}{-3i-3i}=-\frac{1}{6i}
\end{aligned}$$

3. Radius of convergence of f centred at 0 is $R=3$ since f is holomorphic on $D(0,3)$
We can represent f as a power series.

$$\begin{aligned}
f(z) &= \frac{1}{9+z^2}\\
&= \frac{1}{9\left(1+\frac{z^2}{9}\right)}\\
&= \frac{1}{9\left(1-\frac{-z^2}{9}\right)}
\end{aligned}$$

and we recognise the geometric series

$$\frac{1}{9}\sum_{n=0}^{\infty}\left(-\frac{z^2}{9}\right)^n = \frac{1}{9}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{9^n}$$

with

$$a_n = \frac{(-1)^n}{9^{n+1}}$$

This is a geometric series as long as

$$\left|\frac{z^2}{9}\right| < 1 \iff |z| < 3$$

Which confirms our result.
Note that we didn't need to use the definition for the power series of f given as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } a_n = \frac{f^{(n)}(0)}{n!}$$

4. Radius of convergence of f centred at i is $R = 2$ since f is holomorphic on $D(i, 2)$.

5.

$$\int_{-\infty}^{\infty} \frac{1}{9+x^2}dx$$

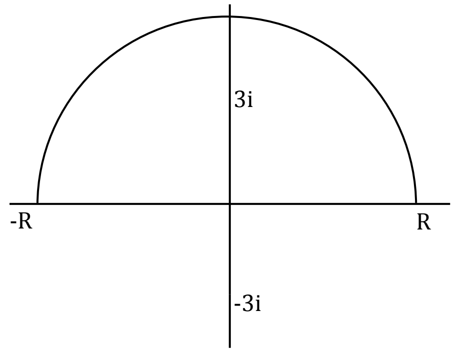
First of all, note that we can calculate this integral using techniques from Analysis 2/ Methods 1. Let's try it this way:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{9+x^2}dx &= \lim_{L\to-\infty, M\to\infty}\int_{L}^{M} \frac{1}{9+x^2}dx \\ &= \lim_{L\to-\infty, M\to\infty}\frac{1}{3}\left[\arctan\left(\frac{x}{3}\right)\right]_L^M = \frac{1}{3}\left(\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right) = \frac{\pi}{3}\end{aligned}$$

Now we are going to solve the same integral with complex analysis, using the same techniques as in the previous week but this time the integral on the contour γ that we pick won't be 0, because the function we are going to choose to integrate will have singularities. So start with the function

$$f(z) = \frac{1}{9+z^2}$$

Take the contour to be (as in the picture) the upper semi-circle from $-R$ to R. Now, since we are taking just the upper semi-circle, only some of the poles will lie in it.



The two poles are at $3i$ and $-3i$, and only the one at $3i$ lies within our contour. So we can calculate the integral on γ with the Residue Formula. It is

$$\begin{aligned}\int_\gamma \frac{1}{9+z^2}dz &= 2\pi i.res(f,3i)\\ &= 2\pi i.\frac{1}{6i}\\ &= \frac{\pi}{3}\text{ , as expected}\end{aligned}$$

And now let's compare this with the integrals along the curves out of which γ is made, i.e.
1. The line segment $[-R,R]$ parametrised by $z(x)=x$ for $-R\le x\le R$.
2. The semi-circular curve γ_R parametrised by $z(t)=Re^{it}$ for $0\le t\le\pi$
So we have

$$\int_\gamma f(z)dz = \int_{\gamma_R} f(z)dz + \int_{[-R,R]} f(z)dz$$

Let's first calculate the second integral on the RHS. We have the parametrisation $z(x)=x$, so $z'(x)=1$ and thus $f(z(x)).z'(x)=f(x)$. So we have

$$\begin{aligned}\int_{[-R,R]} f(z)dz &= \int_{-R}^{R} f(x)dx\\ &= \int_{-R}^{R}\frac{1}{9+x^2}dx\\ &\to \int_{-\infty}^{\infty}\frac{1}{9+x^2}dx \text{ as } R\to\infty\end{aligned}$$

And the other integral is

$$\begin{aligned}\left|.\int_{\gamma_R}\frac{1}{9+z^2}dz\right| &\le (\pi R).\max_{z\in\gamma_R}\left|\frac{1}{9+z^2}\right|\\ &= (\pi R)\max_{|z|=R,y\ge 0}\left|\frac{1}{9+z^2}\right| \quad \le (\pi R)\frac{1}{R^2-9}\to 0 \text{ as } R\to\infty\end{aligned}$$

We got from to the last line by applying the triangle inequality to

$$|9 + z^2| \geq |z^2| - |9| = R^2 - 9$$

So the contribution from this line integral is 0 and thus we get the result we expected.

6.16 Liouville's Theorem: Generalisation

We saw an application of Liouville's Theorem: If f is entire and bounded, then

$$|f(z)| \leq M|z|^{1/2} \text{ , } \forall z, |z| > R \implies f = const$$

And we said that this is fact true for any fractional power of $|z|$ which is less than 1. Then what happens if we take it to be greater than 1? The following example generalised Liouville's Theorem.

Example 61
Let f be entire and assume that for some $M > 0$, $N \in \mathbb{N}$,

$$|f(z)| \leq M(1 + |z|)^N, \forall z \in \mathbb{C} \implies f \text{ is a polynomial of degree less than or equal to N}$$

Proof. Since f is entire, it has a power series at 0 with an infinite radius of convergence.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with

$$a_n = \frac{f^{(n)}(0)}{n!}$$

So

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_N z^N + a_{N+1} z^{N+1} + \ldots\ldots$$

Our aim will be to show that all the terms after $a_N z^N$ are 0, i.e. $a_n = 0$ $\forall n > N$. So consider the following

$$\begin{aligned} |a_n| = \left| \frac{f^{(n)}(0)}{n!} \right| &\leq \frac{\max_{|z|=R} |f(z)|}{R^n} \text{ , by Cauchy's Inequality} \\ &\leq \frac{\max_{|z|=R} M(1 + |z|)^N}{R^n} = \frac{M(1 + R)^N}{R^n} \end{aligned}$$

Now what happens to $\frac{M(1+R)^N}{R^n}$ as $R \to \infty$? N is a fixed number s.t. $n > N$, so the denominator tends to infinity faster that the numerator. So the whole expression tends to 0 and thus we have proved that

$$a_n = 0, \forall n > N$$

Therefore, f is a polynomial of degree less than or equal to N. $\square$

6.17 (T) The Principle of Analytic Continuation

Theorem 31 (The Principle of Analytic Continuation)
Let f be holomorphic in a region Ω, and let w_k for $k = 1, 2, 3, \dots$ and $z_0 \in \Omega$, with $z_o \neq w_k$, $\forall k \in \mathbb{N}$ and

$$\lim_{k\to\infty} w_k = z_0$$

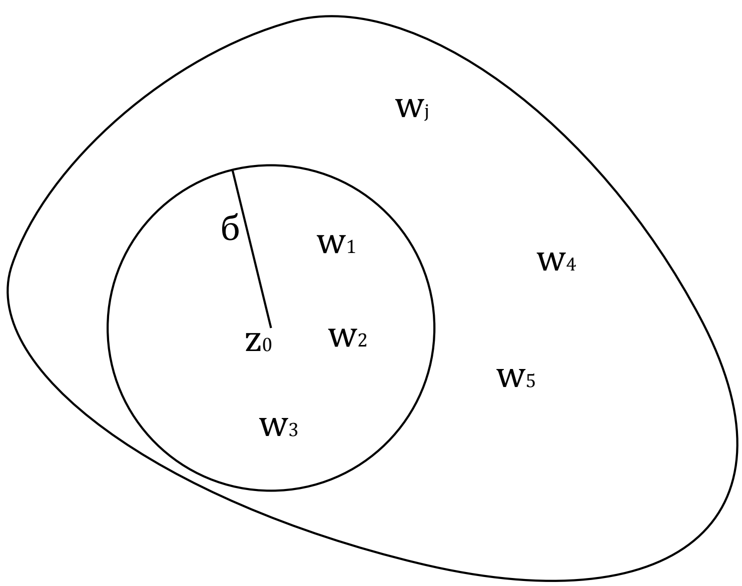
(Such a point z_0 is called *a limit point* in the region)
Moreover, we assume that $f(w_k) = 0$, $\forall k \in \mathbb{N}$.
Then

$$f(z) = 0, \forall z \in \Omega$$

Proof. f is holomorphic at $z_0 \in \Omega$, so f has a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

and it converges inside the radius of convergence R, i.e. for $|z - z_0| < R$.
Assume that $f(z)$ is not identically 0 on the disk $|z - z_0| < R$. We have seen that there exists a minimum N s.t. $a_N \neq 0$ and then $f(z) = (z - z_0)^N g(z)$, with $g(z)$ holomorphic and not-vanishing on the disk $D(z_0, \delta)$, so f cannot have other zeroes on the disk $D(z_0, \delta)$ apart from the z_0. (which we get from $(z - z_0)^N = 0 \iff z - z_0 = 0 \iff z = z_0$).
But then we have the following picture:



While $f(z) \neq 0$ on the disk $D(z_0, \delta) - \{z_0\}$, contained in out region Ω, and z_0 is the single zero, there are also other zeroes - some of the w_j's. So z_0 is not in fact the only zero in the disk. The argument written rigorously reads:
This is contradiction as $w_k \to z_0$, so we can find a $w_k \in D(z_0, \delta)$ with $w_k \neq z_0$ and $f(w_k) = 0$

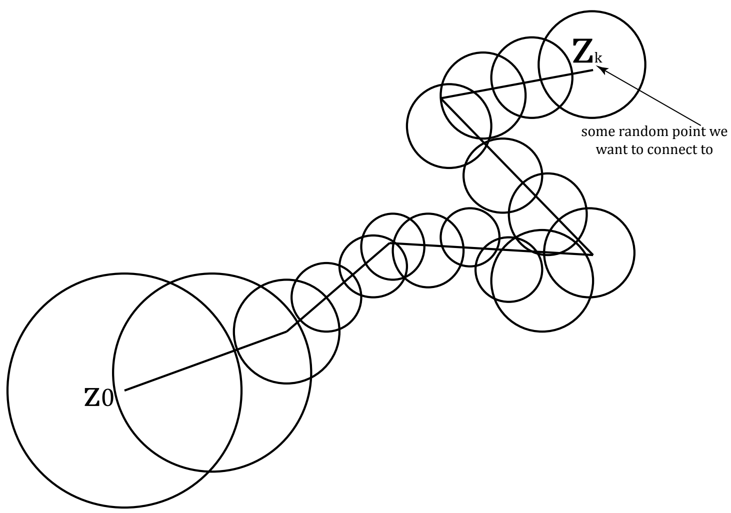
by assumption. This implies that in fact $f(z) = 0$, $\forall z \in D(z_0, \delta)$.
Thus we have proved by contradiction that f is identically 0 in a disk centred at z_0.

$$f(z) = \sum_{n=0}^{\infty} b_n |z - z_0|^n$$

where

$$b_n = \frac{f^{(n)}(z_0)}{n!} = 0$$

since f is zero on $D(z_0, \delta)$.
To prove that f is identically 0 everywhere in the region Ω, we do the following: We pick a point z_1 which lies inside the disk $D(z_0, \delta)$ close to the boundary and we draw a disk around z_1 or radius δ_1. Then f is identically zero on this new disk. We then pick a point z_2 inside the disk $D(z_1, \delta_1)$ close to the boundary and draw a circle around it. Then f is identically 0 on this new disk. If we continue in this way, we can connect the point z_0 to any point inside the region Ω and thus show that f is identically 0 in the whole region.



□

Corollary 11
If f and g are holomorphic on a region Ω and $f(z) = g(z)$, $\forall z \in D(a, \delta) \subset \Omega$. (or even $f(w_k) = g(w_k)$, $\forall k \in \mathbb{N}$), with $< w_k >$ having a limit point $z_0 \in \Omega$, then

$$f(z) = g(z) \text{ in the whole of } \Omega$$

The following is just a random picture created with a basic tool from Adobe Illustrator which I found out about while drawing the diagrams for the previous theorem. I think it's cute!

Example 62

$$f(z) = \sum_{n=0}^{\infty} z^n, |z| < 1$$

Let $\Omega = \{z, |z| < 1\}$. f is holomorphic on Ω
Now, the function

$$g(z) = \frac{1}{1-z}$$

is holomorpic everywhere except for 1, so on $\Omega' = \{z \in \mathbb{C} - \{1\}\}$
But the functions f and g are connected, since

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \text{ for } |z| < 1$$

So for

$$f : \Omega \mapsto \mathbb{C}$$

$$g : \Omega' \mapsto \mathbb{C}$$

with $\Omega \subset \Omega'$, we say that g is the analytic continuation of f from the smaller domain Ω to the larger domain Ω'.

Example 63
Does there exist a holomorphic function on $D(0,1)$ with

$$f\left(\frac{1}{n}\right) = (-1)^n, n \in \mathbb{N}$$

No. f is not even continuous on this disk. Consider

$$\lim_{nt\ \infty} f\left(\frac{1}{n}\right) = f(0) \text{ , since } \lim_{n\to\infty} \frac{1}{n} 0$$

So

$$\lim_{n\to\infty} (-1)^n = f(0)$$

But this limit doesn't even exists. f is not continuous and thus certainly not holomorphic.

Example 64
Find all holomophic functions on $D(0,1)$ with

$$f\left(\frac{1}{n}\right) = \frac{n}{n+1}$$

There exists at most one such function. If there existed another such function then

$$f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right), \forall n \in \mathbb{N}$$

But $\frac{1}{n} \to 0$, $\frac{1}{n} \neq 0$; 0 is a limit point on $D(0,1)$. Therefore, $f(z) = g(z)\ \forall z \in D(0,1)$ by the corollary of The Principle of Analytic Continuation
To find the function, we set $z = \frac{1}{n} \iff n = \frac{1}{z}$ and we get

$$\begin{aligned} f(z) = \frac{\frac{1}{z}}{\frac{1}{z}+1} &= \frac{\frac{1}{z}}{\frac{1+z}{z}} \\ &= \frac{1}{z+1} \end{aligned}$$

Example 65
The power series

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

has radius of convergence 1, so it represents a holomorphic function in the disk $D(0,1)$. We recognise this power series as

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

As in a previous example, here log is holomorphic in a bigger domain that the one of the power series. The function requires a cut from $-\infty$ to -1, but is continuous everywhere else. i.e. on $\mathbb{C} - \{x, x \leq -1\}$, $\log(1+z)$ is holomorphic.
Thus $log(1+z)$ is the analytic continuation of the power series.
What about the power series expansion around $z_0 = 1$? We'd need to work them out from first principles by using

$$f(z_0) = f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \frac{f'''(z_0)}{3!}(z-z_0)^3 + \dots$$

Where

$$\begin{aligned} f(1) &= \log(2) \\ f'(1) &= \frac{1}{1+z}(1) = \frac{1}{2} \\ f''(1) &= \frac{-1}{(1+z)^2}(1) = \frac{-1}{4} \\ f'''(1) &= \frac{2}{8} \\ f^{(n)}(1) &= (-1)^{n-1}\frac{(n-1)!}{(1+n)^n} = (-1)^{n-1}\frac{(n-1)!}{2^n} \end{aligned}$$

so around the point $z_0 = 1$,

$$\log(1+z) = \log(2) + \sum_{n \geq 1} \frac{(-1)^{n-1}(n-1)^n}{2^n.n}$$

6.18 (M) Solving integrals of the type $\int_0^\infty \frac{p(x)}{q(x)}dx$

Method 2 (Solving integrals)
of the type

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$$

or

$$\int_{0}^{\infty} \frac{p(x)}{q(x)}dx$$

where $p(x)$ and $q(x)$ are real polynomials with no common factors and $q(x)$ has no real roots.
We use the upper or lower semi-circular contour, traversed in the positive sense. The only residues we need to consider are the ones inside the contour, so generally only half of them, if the function is even. This saves work when calculating the residues and applying the Residue Theorem.

Example 66
Calculate

$$\int_{0}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}dx$$

First of all, this is an even function, so it is in fact equal to

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}dx$$

and this is the function we are going to aim to get.
Let's work with the upper semi-circular contour and the function

$$f(z) = \frac{2z^2-1}{z^4+5z^2+4}$$

First we need to find the poles of the function. These occur when the denominator is 0, so

$$\begin{aligned} z^4+5z^2+4 &= (z^2+1)(z^2+4) \\ &= (z-i)(z+i)(z-2i)(z+2i) \\ &\therefore z = i, -i, 2i, -2i \end{aligned}$$

In fact the only poles that lie within the region we are considering are i and $2i$.

Next we calculate the residues:

$$\begin{aligned}
Res(f,i) &= \lim_{z\to i}(z-i)f(z) \\
&= \lim_{z\to i}(z-i)\frac{2x^2-1}{(z-i)(z+i)(z-2i)(z+2i)} \\
&= \lim_{z\to i}\frac{2z^2-1}{(z+i)(z^2+4)} \\
&= \frac{2i-1}{(i+1)(-1+4)} \\
&= \frac{-3}{2i+3} = \frac{-1}{2i}
\end{aligned}$$

$$\begin{aligned}
Res(f,2i) &= \lim_{z\to 2i}(z-2i)f(z) \\
&= \lim_{z\to 2i}(z-2i)\frac{2z^2-1}{(z-2i)(z+2i)(z^2+1)} \\
&= \lim_{z\to 2i}\frac{2z^2-1}{(z+2i)(z^2+1)} \\
&= \frac{-8-1}{(4i)(-4+1)} = \frac{3}{4i}
\end{aligned}$$

Then, by the Residue Theorem, we have

$$\begin{aligned}
\int_\gamma f(z)dz &= 2\pi i\left(Res(f,i)+Res(f,2i)\right) \\
&= 2\pi i\left(\frac{-1}{2i}+\frac{3}{4i}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

Now, as usual, we evaluate the same integral by splitting the contour into pieces, integrating and then comparing results.
The contour γ consists of the line segment $[-R,R]$, parametrised by $z(x)=x$, $-R<x<R$. and the semi-circle given by γ_R, parametrised by $z(t)=Re^{it}$, $0<t<\pi$
Let's first evaluate the line segment integral:

$$\begin{aligned}
\int_{[-R,R]} f(z)dz &= \int_{-R}^{R} f(x)dx \\
&= \int_{-R}^{R}\frac{2x^2-1}{x^4+5x^2+4}dx \\
&\to \int_{-\infty}^{\infty}\frac{2x^2-1}{x^4+5x^2+4}dx \text{ ,as } R\to\infty
\end{aligned}$$

This is what we want ot get in the end, so we expect the contribution from the other integral to be 0. We are going to prove this by considering the modulus of the integral:

$$\begin{aligned}
\left| \int_{\gamma_R} f(z)dz \right| &\leq \pi R \max_{|z|=R} |f(z)| \\
&= \pi R \max_{|z|=R} \frac{2z^2-1}{z^4+5z^2+4} \\
&\leq \pi R \left(\frac{|2z^2|+1}{|(z^2+1)(z^2+4)|} \right) \text{from the Triangle Inequality} \\
&= \pi R \left(\frac{2R^2+1}{|(z^2+1)(z^2+4)|} \right)
\end{aligned}$$

For the final step, we need the Triangle Inequality again

$$|z^2+1| \geq |z^2| - 1 = R^2 - 1$$

$$|z^2+4| \geq |z^2| - 4 = R^2 - 4$$

So we get

$$\left| \int_{\gamma_R} f(z)dz \right| \leq \frac{2R^2+1}{(R^2-1)(R^2-4)} \to 0 \text{ as } R \to \infty$$

So the integral along γ_R is indeed 0.
So we end up with

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4} dx = \frac{\pi}{2}$$

And so

$$\int_{0}^{\infty} \frac{2x^2-1}{x^4+5x^2+4} dx = \pi$$

6.19 (M) Solving integrals of the type $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)dx$ or $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)dx$

Method 3 (Solving integrals)
of the type

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)dx$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)dx$$

For $a > 0$, we use the upper semi-circular contour and the function

$$f(z) = \frac{p(z)}{q(z)} e^{iaz}$$

For $a < 0$, we use the lower semi-circular contour and the same function.
The thing which is useful in choosing this function is that when we consider the modulus of the integral along the semi-circular arch, and when we bound it above, we can use the fact that

$$\begin{aligned} |e^{iaz}| &= |e^{ia(x+iy)}| = |e^{iax}.e^{-ay}| \\ &= |e^{iax}||e^{-ay}| = 1.|e^{-ay}| \leq 1 \text{ since } a > 0, y \geq 0 \end{aligned}$$

Example 67
Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx = \frac{\pi}{e}$$

We consider the function

$$f(z) = \frac{e^{iz}}{(z^2+1)^2}$$

and the semi-circular contour from $-R$ to$R$, where the arch is called γ_R.
The function has poles at $z^2 + 1 = 0 \iff z = i, -i$, but only one of them is inside the region we are considering, so we only need to compute the residue at i. However, this is a double pole of f, so we need to apply the appropriate formula when computing the residue. We have

$$\begin{aligned} res(f,i) &= \lim_{z \to i} \frac{d}{dz} (z-i)^2 f(z) dz \\ &= \lim_{z \to i} \frac{d}{dz} \frac{e^{iz}}{(z+i)^2} \\ &= \lim_{z \to i} \frac{ie^{iz}(z+i)^2 - 2(z+i)e^{iz}}{(z+i)^4} \\ &= \frac{e^{ii}(i+i)^2 - 2(i+i)e^{ii}}{(i+i)^4} \\ &= \frac{ie^{-1}(-4) - 4ie^{-1}}{16} \\ &= \frac{-8ie^{-1}}{16} = \frac{-i}{2e} \end{aligned}$$

So we have that, by the Residue Theorem,

$$\int_\gamma \frac{e^{iz}}{(z^2+1)^2} dz = 2\pi i res(f,i) = 2\pi i \frac{-i}{2e} = \frac{\pi}{e}$$

Not we calculate the same integral over the curve γ split into the line segment $[-R, R]$ and the semi-circular arch γ_R, so that

$$\frac{\pi}{e} = \int_{\gamma_R} f(z) dz + \int_{[-R,R]} f(z) dz$$

Consider first the integral of γ_R:

$$\left|\int_{\gamma_r} f(z)dz\right| \leq \pi R \max_{|z|=R} |f(z)|$$

Not, consider just

$$|f(z)| = \left|\frac{e^{iz}}{(z^2+1)^2}\right|$$

We aim to bound the numerator below and the denominator above, so we have

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix-y}| = |e^{ix}||e^{-y}| = |e^{-y}| \leq 1 \text{ , since } y \geq 0$$

$$|z^2+1| \geq |z^2| - 1 = |z|^2 - 1 = R^2 - 1 \text{ on } |z| = R$$

So we have

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \pi R \frac{1}{(R^2-1)^2} \to 0 \text{ as } R \to \infty$$

So the contribution of the integral on the semi-circular arch is 0. Now the one on the line segment:
Let $z(x) = x$ s.t. $-R \leq x \leq R$ and $f(z(x))z'(x) = f(x)$. So we get

$$\begin{aligned}
\int_{[-R,R]} f(z)dz &= \int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2} dx \\
&= \int_{-R}^{R} \frac{\cos x + i \sin x}{(x^2+1)^2} dx \\
&= \int_{-R}^{R} \frac{\cos x}{(x^2+1)^2} dx + i \int_{-R}^{R} \frac{\sin x}{(x^2+1)^2} dx \quad \to \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx + i \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)^2} dx
\end{aligned}$$

So we end up with

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx + i \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)^2} dx = \frac{\pi}{e}$$

Comparing real and imaginary parts gives us

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx = \frac{\pi}{e}$$

and

$$\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)^2} dx = 0$$

where the question didn't ask fo the second part, but we got it anyway. The result isn't surprising as the function we are integrating is odd, so the area under the graph to the right of the x-axis cancels the area to the left of the x-axis.

6.19.1 Jordan's Lemma

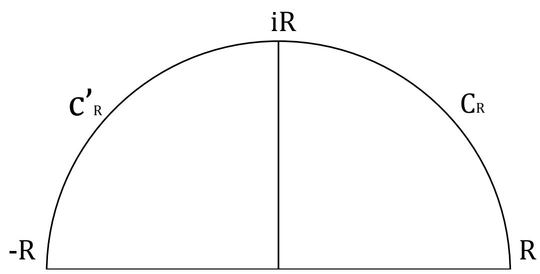
Lemma 4 (Jordan's Lemma)
Let γ_R be the curve $z = Re^{it}$ with $0 \le t \le \pi$.
Suppose that

$$\max_{z \in \gamma_R} |f(z)| \to 0 \text{ as } R \to \infty$$

and take $a > 0$. Then

$$\lim_{R \to \infty} \int_{\gamma_R} e^{iz} f(z) dz = 0$$

Proof. Consider the segment (the RHS part of it only. We are going to consider the LHS of it later. One diagram just because I was too lazy to make two separate ones)



We use the parametrisation $z(t) = Re^{it}$, where $0 \le t \le \frac{\pi}{2}$ and $z'(t) = iRe^{it}$. We have

$$\begin{aligned}
\left| \int_{C_R} e^{iz} f(z) dz \right| &= \int_0^{\pi/2} e^{iaRe^{it}} f(Re^{it}) iRe^{it} dt \\
&\le \int_0^{\pi/2} \left| e^{iaRe^{it}} f(Re^{it}) iRe^{it} \right| dt \\
&\le \int_0^{\pi/2} \left| e^{-aR\frac{2}{\pi}t} f(Re^{it}) R \right| dt
\end{aligned}$$

where we got to the last line from the previous one using

$$\begin{aligned}
|e^{iaz}| &= |e^{iaRe^{it}}| = |e^{iaR\cos t - aR\sin t}| \\
&= |e^{iaR\cos t}||e^{-aR\sin t}| = 1.|e^{-aR\sin t}| \le e^{-aR\frac{2t}{\pi}} \text{ as } a > 0
\end{aligned}$$

So we've got

$$\begin{aligned}
\int_0^{\pi/2} \left|e^{-aR\frac{2}{\pi}t} f(Re^{it})R\right| dt &\le R \max_{z\in\gamma_R} |f(z)| \int_0^{\pi/2} e^{-aR\frac{2t}{\pi}} dt \\
&= R \max_{z\in\gamma_R} |f(z)| \left[\frac{e^{-aR\frac{2t}{\pi}}}{-aR\frac{2}{\pi}}\right]_0^{\frac{\pi}{2}} \text{ after integrating the above} \\
&= \max_{z\in\gamma_R} |f(z)| \left(\frac{\pi}{2a} - \frac{\pi}{2a}e^{-aR}\right) \\
&= \frac{\pi}{2a} \max_{z\in\gamma_R} |f(z)| (1 - e^{-aR}) \to 0 \text{ as } R \to \infty \text{ since} \\
&\max_{z\in\gamma_R} |f(z)| \to \infty \text{ as } R \to \infty \text{ by the statement of the teorem}
\end{aligned}$$

To prove the lemma, it remains to prove that the integral along the other part of the curve also goes to 0 as $R \to \infty$. But this integral is

$$\begin{aligned}
\left|\int_{C_R'} e^{iaz} f(z) dz\right| &\le \int_{\pi/2}^{\pi} e^{-aR\sin t} R |f(Re^{it})| dt \\
&\le R \max_{z\in\gamma_R} |f(z)| \int_{\pi/2}^{\pi} e^{-aR\sin t} dt \\
&= R \max_{z\in\gamma_R} |f(z)| \int_0^{\pi/2} e^{-aR\sin t} dt
\end{aligned}$$

since

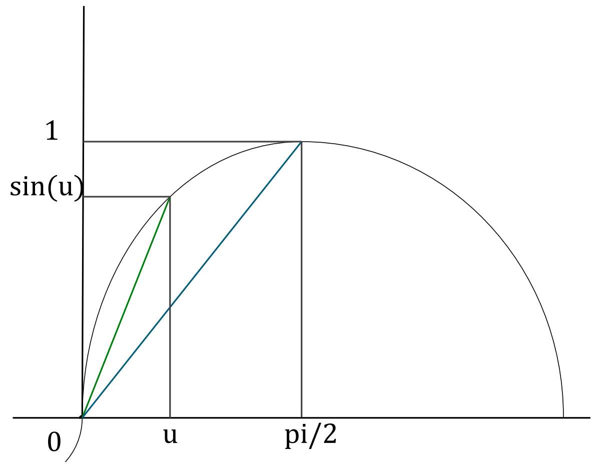
$$\int_0^{\pi/2} e^{-aR\sin t} dt = \int_{\pi/2}^{\pi} e^{-aR\sin t} dt$$

by symmetry The proof is identical afterwards. □

Example 68
Claim:

$$\sin(u) \ge \frac{2}{\pi} u \text{ for } 0 < u \le \frac{\pi}{2}$$

We proved this with differentiation in a homework. Here is another argument:



Since $(\sin u)'' = -\sin u \le 0$ on $0 < u \le \frac{\pi}{2}$, $g(u) = sin(u)$ is concave downwards.
We draw the following graph with two secant lines: Slope of the green secant segment $= \frac{\sin(u)-\sin(0)}{u-0} = \frac{\sin(u)}{u}$
Slope of the blue secant segment $= \frac{1-0}{0-\frac{\pi}{2}} = \frac{2}{\pi}$
Since the first slope is obviously bigger than the second one, we have

$$\frac{\sin(u)}{u} \ge \frac{2}{\pi} \implies \sin(u) \ge \frac{2u}{\pi}$$

6.20 (M) Solving integrals of the type $\int\limits_0^{2\pi} F(\sin t, \cos t)dt$

Method 4 (Solving integrals)
of the type

$$\int\limits_0^{2\pi} F(\sin t, \cos t)dt$$

We use the unit circle as our contour and take the parametrisation

$$z = e^{it}, 0 \le t \le 2\pi$$

Then

$$z' = ie^{it}, \text{ so } dz = ie^{it}dt = izdt$$

And we write

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{z + z^{-1}}{2}$$
$$\sin t = \frac{e^{it} - e^{-it}}{2i} = \frac{z - z^{-1}}{2i}$$

So we end up with

$$\int\limits_0^{2\pi} F(\sin t, \cos t)dt = \int\limits_{|z|=1} F\left(\left(\frac{z - z^{-1}}{2i}\right), \left(\frac{z + z^{-1}}{2}\right)\right)\frac{dz}{iz}$$

Example 69
Prove that

$$\int_0^{2\pi} \frac{dt}{1+a\sin t} = \frac{2\pi}{\sqrt{(1-a^2)}}\ , \text{ where } -1 < a < 1$$

Taking

$$a\sin t = a\frac{z-z^{-1}}{2i}$$

we get, using the method above

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{1+a\sin t}dt &= \int_{|z|=1} \frac{1}{1+a\frac{z-z^{-1}}{2i}}\frac{dz}{iz} \\
&= \int_{|z|=1} \frac{2i}{2i+a(z-z^{-1})}\frac{dz}{iz} \\
&= \int_{|z|=1} \frac{2dz}{2iz+az^2-a} \\
&= \int_{|z|=1} \frac{2}{az^2+2iz-a}
\end{aligned}$$

We have different cases for a in the given interval. If $a=0$, then the problem reduces to

$$\int_{|z|=1} \frac{2dz}{2iz} = \int_{|z|=1} \frac{1}{z}dz = 2\pi i\frac{2}{2i} = 2\pi$$

If $a \neq 0$, then we are back to the integral again and we solve it as usual with the Residue Theorem, but first we need to find the poles which lie inside our contour, the unit circle. The poles are

$$\begin{aligned}
az^2+2iz-a=0 \iff z_{1,2} &= \frac{-2i+/-\sqrt{(-4-4a(-a))}}{2a} \\
&= \frac{-2i+/-\sqrt{-4+4a^2}}{2a} = \frac{-i+/-\sqrt{-1+a^2}}{a} \\
&= \frac{-i+/-i\sqrt{1-a^2}}{a} = i\left(\frac{1+/-\sqrt{1-a^2}}{a}\right)
\end{aligned}$$

where we wrote the roots as purely imaginary in the last step, since $-1<a<1$, $a\neq 0$. Which of these 2 roots lie inside the unit circle? We will prove that

$$z_1 = \frac{-i+\sqrt{-1+a^2}}{a}$$

lies inside the unit disk, $|z| \le 1$

$$\begin{aligned}
\left|\frac{-i+\sqrt{-1+a^2}}{a}\right| \le 1 \iff & \frac{|-1+\sqrt{1-a^2}|}{|a|} \le 1 \\
\iff & \frac{1-\sqrt{1-a^2}}{|a|} \le 1 \iff 1-\sqrt{1-a^2} \le |a| \\
\iff & (1+a-a^2) - 2\sqrt{1-a^2} < |a|^2 \iff 1+1-a^2-a^2 < 2\sqrt{1-a^2} \\
\iff & 2(1-a^2) < \sqrt{1-a^2} \iff \sqrt{1-a^2} < 1
\end{aligned}$$

which is true.
Where is the other root? Applying Vieta's formuli on the original equation, we get that $z_1.z_2 = \frac{-a}{a} = -1$. So the product of the roots is -1 and so the modulus of the product is 1. We can use this fact to determine if the other root lies inside the unit circle.

$$|z_1 z_2| = 1 \implies |z_2| = \frac{1}{|z_1|}$$

But $|z_1| < 1$ and so

$$|z_2| = \frac{1}{|z_1|} > 1$$

so the other root is outside the unit circle. We can now compute the residue of the simple pole z_1. Note that we can factorise f as

$$f = \frac{2}{a(z-z_1)(z-z_2)}$$

$$\begin{aligned}
res(f, z_1) &= \lim_{z\to z_1} (z-z_1)\frac{2}{a(z-z_1)(z-z_2)} \\
&= \lim_{z\to z_1} \frac{2}{a(z-z_2)} \\
&= \frac{2}{a(z_1-z_2)}
\end{aligned}$$

Now

$$z_1 - z_2 = i\left(\frac{-1+\sqrt{1-a^2}}{a} - \frac{-1-\sqrt{1-a^2}}{a}\right) = \frac{2i\sqrt{1-a^2}}{a}$$

So

$$res(f, z_1) = \frac{2}{2i\sqrt{1-a^2}} = \frac{1}{i\sqrt{1-a^2}}$$

Then the Residue Theorem gives

$$\int_{|z|=1} f(z)dz = 2\pi i res(f, z_1) = 2\pi i \frac{1}{i\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}$$

6.21 (T) Laurent Expansion

Theorem 32 (Laurent Expansion)
Let f be holomorphic on $\{z; R_1 < |z - z_0| < R_2\}$ with $0 \leq R_1 < R_2 \leq \infty$. Then we have an expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \text{ on the above set}$$

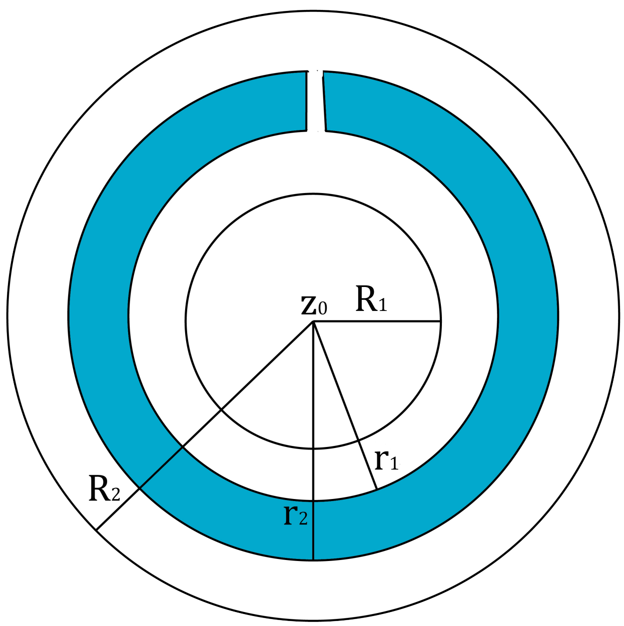
where

$$a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

where $R_1 < r < R_2$

Note that the theorem is very similar to the power series expansion of f, except that in this case, we are working over the annular region and we allow negative powers in $(z - z_0)$ in the sum.
The proof is also very similar to the proof of the Taylor expansion.

Proof. In this proof we are going to include two circles, of radii r_1 and r_2 inside the big circle γ., s.t. the new circles lie between the circle of radius R_1 and the circle of radius R_2, as in the picture below:



As we let the width of the corridor of the keyhole go to 0, the contribution of the integral over the corridors goes to 0.
So, we have

$$R_1 < r_1 < |z - z_0| < r_2 < R_2$$

And now we are going to apply Cauchy's Integral formula of the integral over γ, which is the circle of radius r_2 Note that we are traversing the circle of radius r_2 in the positive sense, and the circle of radius r_1 in the negative sense, since we go from one to the other via the corridors. So we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}dz \\ &= \frac{1}{2\pi i}\int_{|w-z_0|=r_2}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{|w-z_0|=r_1}\frac{f(w)}{w-z}dw \\ &+ 0 \text{ which is the contribution of the corridors} \end{aligned}$$

Consider first $|w-z_0| = r_2$. We have that

$$\left|\frac{z-z_0}{w-z_0}\right| < 1$$

since the w's lie on the circle. Now we have

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{w-z_0+z_0-z} \\ &= \frac{1}{w-z_0}\frac{1}{1+\frac{z_0-z}{w-z_0}} \\ &= \frac{1}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} \\ &= \frac{1}{w-z_0}\sum_{n\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^n \end{aligned}$$

Thus

$$\int_{|z-z_0|=r_2}\frac{f(w)}{w-z}dw = \sum_{n\geq 0}\int_{|w-z_0|=r_2}\frac{f(w)}{(w-z_0)^{n+1}}dw(z-z_0)^n$$

Consider now $|w-z_0| = r_1$, where we have that

$$\left|\frac{w-z_0}{z-z_0}\right| < 1$$

We have

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{w-z_0+z_0-z} \\ &= \frac{-1}{z-z_0}\cdot\frac{1}{1-\frac{w-z_0}{z-z_0}} \\ &= \frac{-1}{z-z_0}\sum_{k\geq 0}\left(\frac{w-z_0}{z-z_0}\right)^k \end{aligned}$$

Thus

$$\int_{|z-z_0|=r_1}\frac{f(w)}{w-z}dw = -\sum_{k\geq 0}\int_{|w-z_0|=r_1} f(w)(w-z_0)^k(z-z_0)^{-(1+k)}dw$$

Set $k+1=-n$, so that $k \geq 0 \iff n<0$, so we can rewrite the above as

$$\sum_{n<0} \int_{|w-z_0|=r_1} \frac{f(w)}{(w-z_0)^{n+1}} dw.(z-z_0)^n$$

So we end up with

$$f(z)=\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}dw=\frac{1}{2\pi i}\left(\sum_{n\geq 0}\int_{|w-z_0|=r_2}\frac{f(w)}{(w-z_0)^{n+1}}dw(z-z_0)^n-\sum_{n<0}\int_{|w-z_0|=r_1}\frac{f(w)}{(w-z_0)^{n+1}}dw(z-z_0)^n\right)$$

The final step is to show that this is valid when integrating over $C_r(z_0)$, where $R_1<r<R_2$ By Cauchy's Theorem, as long as the function we are integrating is holomorphic (which it is in this case), we have

$$\int_{C_r(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw-\int_{C_{r_1}(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw=0 \implies \int_{C_r(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw=\int_{C_{r_1}(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw$$

$$\int_{C_r(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw-\int_{C_{r_2}(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw=0 \implies \int_{C_r(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw=\int_{C_{r_2}(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw$$

Where r is greater than r_1 and r_2, but still less than R_2.
So now we have

$$f(z)=\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}dw=\frac{1}{2\pi i}\left(\sum_{n\geq 0}\int_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{n+1}}dw(z-z_0)^n-\sum_{n<0}\int_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{n+1}}dw(z-z_0)^n\right)$$

Which is the same as

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n \text{ on the above set}$$

where

$$a_n=\frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw$$

$\square$

Example 70
Find the Laurent expansion of

$$g(z)=\frac{z}{(1+z)(1-z)}$$

for

1. $|z|<1$

2. $|z-1| > 2$

Solution: *Part 1*

In $|z| < 1$, the function doesn't have a vanishing denominator, so g is holomorphic on $|z| < 1$ and so we write the Taylor series. We need to write it in the form

$$g(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } a_n = \frac{1}{2\pi i} \int_{C_1(0)} \frac{f(w)}{(w-0)^{n+1}} dw = \frac{g^{(n)}(0)}{n!}$$

But we don't actually need to work from these definitions (in general), since we can just consider the geometric series. We are first going to split the function into partial fractions:

$$\frac{A}{1+z} + \frac{B}{1-z} = \frac{z}{(1+z)(1-z)}$$

We find that $A = \frac{-1}{2}$ and $B = \frac{1}{2}$. So we have

$$g(z) = -\frac{1}{2}\left(\frac{1}{1+z}\right) + \frac{1}{2}\left(\frac{1}{1-z}\right)$$

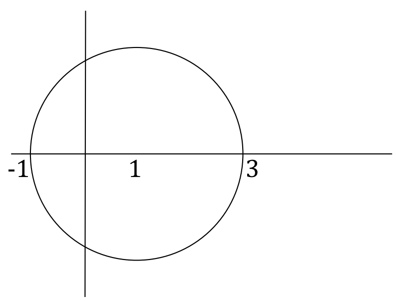
which we can write as a geometric series

$$g(z) = -\frac{1}{2}\sum_{n=0}^{\infty}(-z)^n + \frac{1}{2}\sum_{n=0}^{\infty}(-z)^n = \frac{1}{2}\sum_{n=0}^{\infty}\left(-(-z)^n + (z)^n\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left((-1)^{n+1}+1\right) z^n$$

and this is the final answer.

Solution: *Part 2*

We need to find the expansion outside the disk $|z-1| > 2$



The centre of the disk is at 1, so the Laurent expansion should be of the form

$$g(z) = \sum_{n=-\infty}^{\infty} a_n (z-1)^n$$

So we need to express the function in terms on $(z-1)$ before doing anything else to it.

The condition $|z-1| > 2$ means that $\frac{2}{|z-1|} < 1$.

Consider first $\frac{1}{1+z}$. We can write it as

$$\frac{1}{1+z} = \frac{1}{2+z-1} = \frac{1}{(z-1)\left(\frac{2}{z-1}+1\right)}$$

Thus we can write

$$\begin{aligned}
g(z) &= \frac{z}{(1+z)(1-z)} = \frac{z-1+1}{(z-1)\left(\frac{2}{z-1}+1\right)(1-z)} \\
&= \frac{(z-1)}{(z-1)\left(\frac{2}{z-1}+1\right)(1-z)} + \frac{1}{(z-1)\left(\frac{2}{z-1}+1\right)(1-z)} \\
&= \frac{-1}{(z-1)\left(\frac{2}{z-1}+1\right)} + \frac{-1}{(z-1)^2\left(\frac{2}{z-1}+1\right)} \\
&= \frac{(-1)}{z-1}\sum_{n=0}^{\infty}\left(\frac{-2}{z-1}\right)^n - \frac{1}{(z-1)^2}\sum_{n=0}^{\infty}\left(\frac{-2}{z-1}\right)^n \\
&= \sum_{n=0}^{\infty}(-2)^n(-1)(z-1)^{-n-1} - \sum_{k=0}^{\infty}(-2)^k(z-1)^{-k-2}
\end{aligned}$$

Set $-k-2=-n-1 \implies -k=-n-1+2=-n+1 \implies k=n-4$ so we have that

$$g(z)=\sum_{n=0}^{\infty}(-2)^n(-1)(z-1)^{-n-1}-\sum_{n=1}^{\infty}(-2)^{n-1}(z-1)^{-n-1}$$

So need to start from $n=1$. The term for $n=0$ from the first series is $\frac{-1}{z-1}$. So we have

$$g(z)=\sum_{n=1}^{\infty}\left[(-2)^n(-1)-(-2)^{n-1}\right](z-1)^{-n-1}-\frac{-1}{z-1}$$

Note that when we write this out explicitly, the terms in the expansion start at $\frac{1}{1+z}$.

Example 71
Find

$$\int_0^{\infty}\frac{\log x}{(1+x^2)^2}dx$$

Let

$$f(z)=\frac{\log z}{(1+z^2)^2}$$

The contour we are going to consider is the intended upper semi circle. Now, if we took the principle logarithm, it would not be defined on part of our contour - the negative real axis. This is why we don't choose the principle log, but instead we restrict the argument to lie between $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$, i.e. we have chosen the cut for the log to be on the negative imaginary axis. Now we only have one discontinuity to worry about, and this is at the origin, which is why we take the intended semi-circle. So now we have a well-defined contour along which to integrate First we consider the poles of $f(z)$. They occur at

$$(z^2+1)^2=0 \iff (z+i)^2(z-i)^2=0 \iff z=+-i$$

So the points i and $-i$ are poles of order 2. However, only the pole at i lies in our contour. So we only need to find the residue there. We have

$$\begin{aligned}
res(f,i) &= \lim_{z\to i}\frac{d}{dz}\left((z-i)^2 f(z)\right)\\
&= \lim_{z\to i}\frac{d}{dz}\left(\frac{(\log(z))(z-i)^2}{(z+i)^2(z-i)^2}\right)\\
&= \lim_{z\to i}\frac{d}{dz}\left(\frac{\log(z)}{(z+i)^2}\right)\\
&= \lim_{z\to i}\left(\frac{\frac{1}{z}(z+i)^2-2\log(z)(z+i)}{(z+i)^4}\right)\\
&= \left(\frac{\frac{1}{i}(i+i)^2-2\log(i)}{(i+i)^4}\right)\\
&= \frac{\frac{1}{i}2i-2\log(i)}{(2i)^3}\\
&= \frac{2-2\log(i)}{8(-i)}\\
&= \frac{2-2\log|i|-2arg(i)}{-8i}\\
&= \frac{2-i\pi}{-8i}
\end{aligned}$$

So by the Residue Theorem, we have that

$$\int_\gamma f(z)dz = 2\pi i\left(\frac{2-i\pi}{-8i}\right) = -\pi\frac{2-i\pi}{4}$$

Again we are going to evaluate the same integral again by integrating along the contour we have chosen. We split the contour into 4 parts:

- the line segment $[-R,-r]$ parametrised by $z(x)=x$ for $-R\le x\le -r$
- the line segment $[r,R]$ parametrised by $z(x)=x$ for $r\le x\le R$
- the semi-circle γ_R
- the semi-circle γ_r

Let's look first at the integral around the segment $[r,R]$. We have

$$\int_{[r,R]} f(z)dz = \int_r^R \frac{\log(x)}{(1+x^2)^2}dx$$

and this is ok as it is. Let's now look at the integral on the other line segment. It's important to note that on the other segment, we are working on the negative real axis, which means that $\log(x)$ is not the usual real logarithm, but it is actually a complex logarithm, given by

$$\log(x) = \log|x| + i\pi$$

Since the argument on the negative real axis is π. Here $|x|$ is negative, s we leave it a modulus. So now let's consider the integral with its parametrisation:

$$\int_{[-R,-r]} f(z)dz = \int_{-R}^{-r} \frac{\log|x| + i\pi}{(1+x^2)^2}dx = \int_{-R}^{-r} \frac{\log|x|}{(1+x^2)^2}dx + i\pi \int_{-R}^{-r} \frac{1}{(1+x^2)^2}$$

Make the substitution $x = -u$, so that u is now a positive number. The limits of integration become, at $x = -R$, $u = R$ and at $x = -r$, $u = r$. Also, $dx = -du$ and so we now have

$$\begin{aligned}\int_{-R}^{-r} \frac{\log|x|}{(x^2+1)^2}dx &= \int_{R}^{r} \frac{\log(u)}{(u^2+1)^2}(-du) \\ &= \int_{r}^{R} \frac{\log(u)}{(u^2+1)^2}du\end{aligned}$$

where the - sign of du cancels with the - from flipping the limits of integration

But u can just be swapped for another variable, so let's pick x, so that we have

$$\int_{r}^{R} \frac{\log(x)}{(x^2+1)^2}$$

So in total, we have from the two integrals over the line segments,

$$\int_{r}^{R} \frac{\log(x)}{(x^2+1)^2}dx + \int_{r} R\frac{\log(x)}{(x^2+1)^2}dx + \int_{-R}^{-r} \frac{i\pi}{(x^2+1)^2}dx$$

We let $R \to \infty$ and $r \to 0$ and we get

$$2\int_{0}^{\infty} \frac{\log(x)}{(x^2+1)^2}dx + i\int_{-\infty}^{0} \frac{\pi}{(1+x^2)^2}dx$$

Now let's look at the integral over ga_R. We have

$$\left|\cdot\int_{\gamma_R} \frac{\log(z)}{(1+z^2)^2}dz\right| \leq \pi R \max_{z\in\gamma_R}\left|\frac{\log(z)}{(1+z^2)^2}\right|$$

We have that

$$\begin{aligned}|\log(z)| &= |log|z| + iarg(z)| \\ &\leq |\log|z|| + |iarg(z)| \\ &\leq log(R) + \pi \text{ on } |z| = R\end{aligned}$$

Here we used the triangle inequality, the fact that the largest argument that z can have on the upper semi-circle is π and that $|i| = 1$. Similarly, we bound the denominator by using the triangle inequality

$$\begin{aligned}|1 + z^2| &\geq |z^2| - 1 \\ &= |z|^2 - 1 = R^2 - 1 \text{ on } |z| = R\end{aligned}$$

Putting this together, we get that

$$\left|\int_{\gamma_R} \frac{\log(z)}{(1+z^2)^2} dz\right| \leq \pi R \left(\frac{\log(R) + \pi}{(R^2-1)^2}\right) \to 0 \text{ as } R \to \infty$$

Where the last limits is obtained by applying L'Hopital's Rule on $R\log(R)$ and then by taking standard limits.
Now for the last integral, over γ_r, we have that

$$\left|\int_{\gamma_r} f(z)dz\right| \leq \pi r \max_{z\in\gamma_r} \left|\frac{\log(z)}{(1+z^2)^2}\right|$$

First of all, we can assume that r is very small, and in particular that it is less than 1 (otherwise the pole at i wouldn't lie in the region we are considering). We bound the numerator in the same way as before

$$|\log(z)| = |\log|z|| + |iagr(z)| \leq |\log(r)| + i\pi \text{ on } |z| = r$$

For the denominator, we use again the triangle inequality, but in this case we don't write

$$|1 + z^2| \geq |z^2| - 1 = r^2 - 1$$

since we are assuming that $r < 1$ and we don't want to bound the modulus of the integral above with a negative number as it wouldn't tell us much about what is happening. So instead we take the other possibility of the triangle inequality. We have that

$$|1 + z^2| \geq 1 - |z^2| = 1 - r^2$$

Substituting this all back we end up with

$$\left|.\int_{\gamma_r} f(z)dz\right| \leq \pi r \frac{\log(r) + \pi}{(1-r^2)^2} \to 0 \text{ as } r \to 0$$

Where the last limit can again be calculated with L'Hopital's Rule.

Since the contributions of the last two integrals are zero as $R \to \infty$ and $r \to 0$, we get overall that

$$2\int_0^\infty \frac{\log(x)}{(1+x^2)^2} dx + i\pi \int_{-\infty}^0 \frac{1}{(x^2+1)^2} dx = \frac{2 - i\pi}{-8i} = \frac{-2\pi}{4} + i\frac{\pi^2}{2}$$

The last step is to compare the real and imaginary parts. We get the answer we want from the real part:

$$2\int_0^\infty \frac{\log(x)}{(x^2+1)^2}dx = \frac{-\pi}{2} \iff \int_0^\infty \frac{\log(x)}{(x^2+1)^2}dx = \frac{-\pi}{4}$$

The Argument Principle *Part 1*
If f is holomorphic and has a zero of order N at a point z_0, then we can write

$$f(z) = (z-z_0)^N g(z)$$

where $g(z)$ is a holomorphic function at some small disk $D(z_0,\delta)$ for $\delta > 0$ and $g(z) \neq 0$ for $z \in D(z_0,\delta)$.
Differentiating this expression for f, we get

$$f'(z) = N(z-z_0)^{N-1}g(z) + (z-z_0)^N g'(z)$$

and dividing through by f, we get that

$$\frac{f'(z)}{f(z)} = \frac{N(z-z_0)^{N-1}g(z) + (z-z_0)^N g(z)}{(z-z_0)^N g(z)} = \frac{N}{z-z_0} + \frac{g'(z)}{g(z)}$$

Here $z \in D(z_0,\delta)$ and the function $\frac{g'(z)}{g(z)}$ is holomorphic on D because $g(z) \neq 0$. Then we can see that z_0 is a pole of order 1 of $\frac{f'(z)}{f(z)}$
It's residue is N, with is the principle part of $\frac{N}{(z-z_0)}$.
Part 2
Suppose that f has a pole of order m at w_0. Then we can write

$$f(z) = (z-w_0)^{-m}h(z)$$

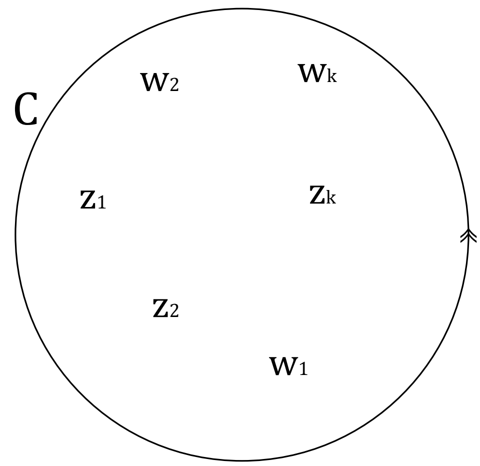
where h is holomorphic on $D(w_0,\delta)$ and $h(z) \neq$ for $z \in D(w_0,\delta)$. Then we have that

$$\frac{f'(z)}{f(z)} = \frac{-m(z-w_0)^{-m-1}h(z) + (z-w_0)^{-m}h'(z)}{(z-w_0)^{-m}h(z)} = \frac{-m}{z-w_0} + \frac{h'(z)}{h(z)}$$

where $\frac{h'(z)}{h(z)}$ is holomorphic as $h(z) \neq 0$. Thus $\frac{f'(z)}{f(z)}$ has a simple pole at w_0 with residue $-m$.
We put this formally into the

Theorem 33 (The Argument Principle)
Suppose that f is holomrphic on an open set containing the toy contour C and its interior with the exception that f can have a finite number of poles at $f(z)$ in its interior.
Suppose that $f(z) \neq 0$ for $z \in C$.
Let N be the number of zeroes of $f(z)$ which are inside C, counted with their multiplicities and let P be the number of poles of f inside C counted with their multiplicities. Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P$$



Proof. We use the Residue Theorem on

$$\frac{f'(z)}{f(z)}$$

This is holomorphic inside C for z not a pole or a zero of $f(z)$.
Let the poles of f be at $w_1, w_2, \ldots, w_s$ and let the zeroes be at $z_1, z_2, \ldots, z_k$.
Then

$$\begin{aligned}\int_C \frac{f'(z)}{f(z)} dz &= 2\pi i \sum \text{ residues} \\ &= \sum \text{ multiplicities of the zeroess of f } - \sum \text{ multiplicities of the poles of f} \\ &= N - P\end{aligned}$$

□

Theorem 34 (Rouche's Theorem)
Suppose that f and g are holomorphic on an open set containing a toy contour C and its interior.
Assume that

$$|f(z)| > |g(z)|, \text{ for } z \in C$$

Then f and $f+g$ have the same number of zeroes inside C.
Note: We are considering zeroes inside C.

Proof.

$$|f(z)| > |g(z)| \geq 0 \text{ for } z \in C \implies f(z) \neq 0 \text{ for } z \in C$$

$$|(f+g)(z)| = |f(z) + g(z)| \geq |f(z)| - |g(z)| > 0 \text{ for } z \in C$$

Therefore,

$$f(z) + g(z) \neq 0 \text{ on } C$$

Call N_{f+g} the number of zeroes of $f+g$. Let N_f be the number of zeroes of f. (not poles. Applying the Argument Principle, we get

$$N_f = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$$

$$N_{f+g}\frac{1}{2\pi i}\int_C \frac{(f+g)'(z)}{(f+g)(z)}dz$$

Then we get that

$$\begin{aligned}
N_{f+g} - N_f &= \frac{1}{2\pi i}\int_C \left(\frac{(f'+g')(z)}{f(z)+g(z)} - \frac{f'(z)}{f(z)}\right)dz \\
&= \frac{1}{2\pi i}\int_C \frac{(f'+g')f-(f+g)f'}{f(f+g)}(z)dz \\
&= \frac{1}{2\pi i}\int_C \frac{g'f-gf'}{f(f+g)}(z)dz
\end{aligned}$$

Now we notice the following

$$\begin{aligned}
\frac{\left(1+\frac{g}{f}\right)'}{1+\frac{f}{g}} &= \frac{\frac{g'f-gf'}{f^2}}{1+\frac{g}{f}} \\
= \frac{(g'f-gf')f}{f^2(f+g)} &= \frac{g'f-gf'}{f(f+g)}
\end{aligned}$$

which is exactly what we are trying to integrate above. So we can write

$$N_{f+g} - N_f = \frac{1}{2\pi i}\int_C \frac{(1+\frac{g}{f})'}{(1+\frac{g}{f})}(z)dz$$

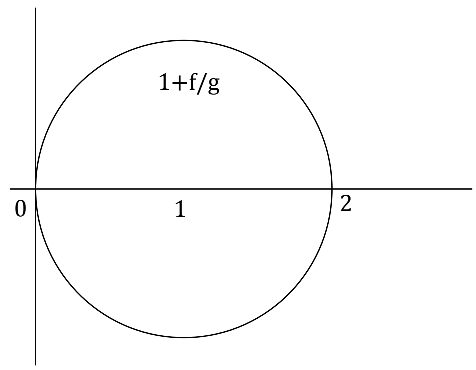
And now we notice that we actually have

$$(\log(1+g/f))' = \frac{(1+g/f)'}{(1+g/f)}$$

so we have found an antiderivative for the integral above and the contour is closed, and hence, by the Antiderivative Theorem, the integral is 0.
It remains only to show that the antiderivative we have found is actually legitimate.
Now, $\log(1+g/f)$ is the composition of the functions $h(z) = 1+g/f$ and $H(z) = log(z)$. We are going to show that the function h never takes value on the negative real axis, so the composition of h and H is a well-defined holomorphic function.
By the statement of the theorem, we have that $|g(z)| < |f(z)|$. We can use this as follows

$$\begin{aligned}
|g(z)| < |f(z)| &\implies \left|\frac{g(z)}{f(z)}\right| < 1 \\
&\implies \left|(1+\frac{g}{f}(z)) - 1\right| < 1 \implies |h-1| < 1
\end{aligned}$$

but this is the equation of the disk centred at 1 on the real axis, with radius 1. So all the values of h are inside this disk. In particular, h doesn't touch the negative real axis. So we can just take the standard cut for the principal logarithm and then the composition of h and H is a well-defined function and it is the antiderivative we want.



$\square$

Example 72
How many zeroes of

$$z^7 - 4z^3 + z - 1 = 0$$

are inside the unit disk? What about the disk or radius 2?
We have, on $|z| = 1$,
$|z^7| = 1$, $|-4z^3| = 4$, $|z| = 1$, $|-1| = 1$
Let

$$f(z) = -4z^3$$
$$g(z) = z^7 + z - 1$$

Then

$$|g(z)| = |z^7 + z - 1| \le |z^7| + |z| + 1 = 1 + 1 + 1 = 3$$
$$|f(z)| = 4$$

so

$$|g(z)| < |f(z)|$$

Therefore, $f(z) = -4z^3$ has the same number of zeroes inside $|z| = 1$ as does $f(z) + g(z) = -4z^3 + z^7 + z - 1$
Now, $f(z) = 0 \iff -4z^3 = 0 \iff z = 0$ with multiplicity 3. Therefore $z^7 - 4z^3 + z - 1$ has 3 zeroes inside $|z| = 1$.
Where are the other 4 roots? Let's try the circle with radius 2:
On $|z| = 2$,
$|z^7| = 128$, $|-4z^3| = 32$, $|z| = 2$, $|-1| = 1$.
Since $128 > 32 + 2 + 1 = 35$, take

$$f(z) = z^7$$
$$g(z) = -4z^3 + z - 1$$

Then

$$|g(z)| = |-4z^3 + z - 1| \leq |-4z^3| + |z| + |-1| = 32 + 2 + 1 = 35$$
$$|f(z)| = |z^7| = 128$$

So $|g(z)| < |f(z)|$ and so f and $f + g$ have the same number of zeroes in side $|z| = 2$. Now

$$f(z) = 0 \iff z^7 = 0 \iff z = 0$$

with multiplicity 7. Therefore

$$f(z) + g(z) = z^7 - 4z^3 + z - 1$$

has the same number of root inside $|z| = 2$ and f, i.e. there are 7 roots.
Thus all the roots of the equation lie inside the disk of radius 2.
Note that this doesn't mean they don't lie in a smaller disk, maybe of radius 1.5.
In fact we can ask ourselves the question: Where are the roots located in the until disk? We know that complex roots occur in conjugate pairs and we have 3 roots in the unit disk. So we are either going to have two complex roots which are conjugates and one real root, or 3 real roots.

Example 73
Show that

$$\int_0^\infty \frac{x^{-a}}{1+x} dx = \frac{\pi}{\sin \pi a} \text{ , for } 0 < a < 1$$

We are going to work with

$$f(z) = \frac{z^{-a}}{1+z}$$

And the following contour: The keyhole contour, which avoids the positive real axis and the origin. In this region, we can take the cut for the logarithm to be on the positive real axis, i.e. we take $0 < arg(z) < 2\pi$. Then the log is holomorphic inside the contour we are working with. Now, we have that

$$e^{\log z} = z$$
$$(e^{\log z})^{-a} = z^{-a} = e^{-a \log z}$$

Our contour consists of the following:
L_1 is the segment from $r + i\delta$ to $R + i\delta$, i.e. the segment $[r + i\delta, R + i\delta]$. We parametrise it by $z(x) = x$ for $r \leq x \leq R$
L_2 is the inverse of L_1, i.e. the segment $[r - i\delta, R - i\delta]$.
γ_R is the bigger circle (the full circle), where we parametrise by $z(t) = Re^{it}$
γ_r is the smaller circle (the full circle) where we parametrise by $z(t) = re^{it}$
Now, computing the integral with the Residue Theorem, we get that

$$res(f, -1) = \lim_{z \to -1} \left((z+1) \frac{z^{-a}}{z+1} \right) = \lim_{z \to -1} e^{-a \log z}$$
$$= e^{-a \log(-1)} = e^{-a \log |-1| + i\pi} = e^{-ai\pi}$$

and so

$$\int_{\gamma} f(z)dz = 2\pi i e^{-ai\pi}$$

Now we are going to consider what happens to our integral on the contour we have chosen, as we let the width of the corridor δ to go to 0.
On L_1, we have that

$$\int_{L_1} f(z)dz \to \int_r^R \frac{e^{-a\log(x)}}{1+x}dx \text{ , as } \delta \to 0$$

this is so, because in this region, the x-values are positive, so the logarithm is the usual real logarithm, so we have that

$$z^{-a} = e^{-a\log(z)} = e^{-a(\log|z|+iarg(z))} \to e^{-a\log(|z|)} = e^{-a\log(x)} \text{ as } \delta \to 0 \text{ when } x \geq 0$$

However, on L_2, x is still positive so we don't get any problems there, but we are approaching the positive real axis from below, so the argument of the logarithm doesn't tend to 0 but to 2π. So we have that

$$z^{-a} = e^{-a\log(z)} = e^{-a(\log|z|+iarg(z))} \to e^{-a(\log(|z|)+i2\pi)} = e^{-a\log(x)}e^{-ai2\pi} \text{ as } \delta \to 0$$

Thus, we get from these two integrals that

$$\int_{L_1} f(z)dz + \int_{L_2} dz = (1-e^{-2\pi i a})\int_r^R \frac{-a\log x}{1+x}$$

Consider now the curve γ_R. We have that

$$\begin{aligned} \left|\int_{\gamma_R} f(z)dz\right| &\leq 2\pi R \max_{z\in\gamma_R}\left|\frac{z^{-a}}{1+z}\right| \\ &\leq 2\pi R \max_{z\in\gamma_R}\frac{|z^{-a}|}{R-1} \end{aligned}$$

since the triangle inequality gives us that $|1+z| \geq |z|-1 = R-1$ on $|z| = R$.
We also have that

$$\begin{aligned} |z^{-a}| &= |e^{-a\log(z)}| = e^{\Re(-a\log z)} = e^{-a\log|z|} \\ &= |z|^{-a} = R^{-a} \text{ on } |z| = R \end{aligned}$$

as $\log(z) = \log|z| + iagr(z)$ and $arg(z)$ is the imaginary part, so $\Re(-a\log(z)) = -a\log|z|$
Thus we have that

$$\left|\int_{\gamma_R} f(z)dz\right| \leq 2\pi R\frac{R^{-a}}{R-1} = 2\pi i\frac{R^{1-a}}{R-1} \to 0 \text{ as } R\to\infty \text{ , since } 0<a<1$$

Finally, consider the integral around γ_e. We have that

$$\left|\int_{\gamma_r} f(z)dz\right| \leq 2\pi r\max_{|z|=r}\left|\frac{z^{-a}}{1+z}\right| \leq 2\pi r\frac{r^{-a}}{1-r} = 2\pi r\frac{r^{-a}}{1-r} = 2\pi r\frac{r^{1-a}}{1-r} \to 0 \text{ as } r\to 0$$

by the same arguments as before, but using the other triangle inequality.
Altogether we have that

$$(1 - e^{-2\pi i a}) \int_r^R \frac{x^{-a}}{1+x} dx = 2\pi i e^{-ai\pi}$$

which becomes, as we let $R \to \infty$ and $r \to 0$

$$(1 - e^{-2\pi i a}) \int_0^\infty \frac{x^{-a}}{1+x} dx = 2\pi i e^{-ai\pi}$$

Therefore,

$$\int_0^\infty \frac{x^{-a}}{1+x} dx = \frac{2\pi i e^{-ai\pi}}{1 - e^{-2\pi i a}} e^{-ai\pi} = \frac{2\pi i}{e^{-ai\pi} - e^{-ai\pi}} = \frac{2\pi i}{2i(\sin \pi a)} = \frac{\pi}{\sin \pi a}$$

Theorem 35 (The Fundamental Theorem of Algebra - Proof with Rouche's Theorem)
Every non-constant polynomial of degree n has n roots, counted with their multiplicities.

Proof.

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

Suppose that $|z| = R$, where R is large. Let

$$f(z) = a_n z^n$$

$$g(z) = a_{n-1} z^{n-1} + \ldots + a_1 z + z_0$$

Then how many zeroes does $f(z)$ have inside of $|z| = R$? We set $f(z) = a_n z^n$, where $a_n \neq 0$, so the zeroes of f are given by

$$a_n z^n = 0 \iff z^n = 0 \iff z = 0 \text{ with multiplicity } n$$

So f has n zeroes. Consider now g. We have that

$$\begin{aligned}
|g(z)| &= |a_{n-1} z^{n-1} + \ldots + a_1 z + a_0| \\
&\leq |a_{n-1} z^{n-1}| + |a_{n-2} z^{n-2}| + \ldots + |a_1 z| + |a_0| \text{ by the triangle inequality} \\
&= |a_{n-1}| R^{n-1} + \ldots + |a_1| R + |a_0| \text{ since here } |z| = R \\
&< |a_n| R^n = |f(z)|
\end{aligned}$$

This means that, by Rouche's theorem, $f + g$ has the same number of roots as f, which has n roots (counted with their multiplicities).
Now we just need to prove that the last step above was legitimate, i.e. we need to prove that

$$|a_{n-1}| R^{n-1} + \ldots + |a_1| R + |a_0| < |a_n|$$

Dividing though by R^n, we get that

$$|a_{n-1}|\frac{1}{R} + \ldots + |a_0|\frac{1}{R^n} < |a_n|$$

But

$$\lim_{R\to\infty}\left(|a_{n-1}|\frac{1}{R} + \ldots + |a_0|\frac{1}{R^n}\right) = 0$$

and therefore, by the Inertia Principle, we have that, for R large enough,

$$|a_{n-1}|\frac{1}{R} + |a_{n-2}|\frac{1}{R^2} + \ldots + |a_0|\frac{1}{R^n} < |a_n|$$

but since we started with a large R, this result definitely holds. This proves the last step we needed to justify and thus the proof of the theorem is complete. □

Theorem 36 (Riemann's Theorem of Removable Singularities)
Let Ω be an open set and let $f : \Omega - \{z_0\} \to \mathbb{C}$ be holomorphic.
Assume that f is bounded on $\Omega - \{z_0\}$.
Then f has a removable singularity at z_0.

Proof. Since f is holomorphic of the annulus $0 < |z - z_0| < R$, we have a Laurent expansion for $f(z)$. That is, $\forall z$ with $0 < |z - z_0| < R$, we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

with

$$a_n = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}}dw$$

Our aim will be to show that for $n < 0$, $a_n = 0$, and thus deduce that the Laurent expansion actually starts from 0 rather than from $-\infty$
We have that f is bounded. Therefore, we can find an M such that $|f(w)| \le M$, $\forall w \in \Omega$.
Fix $n < 0$. Then

$$\begin{aligned}|a_n| &= \left|\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}}dw\right| \\ &\le \frac{1}{2\pi}.2\pi r \max_{z\in C_r(z_0)} \frac{|f(w)|}{|w - z_0|^{n+1}} \\ &\le r.\frac{M}{r^{n+1}} = \frac{M}{r^n} \to 0 \text{ for } r \to 0 \text{ since } -n > 0\end{aligned}$$

Therefore, $a_n = 0$ for $n < 0$.
So for the Laurent series, we get that

$$\sum_0^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + \ldots$$

and this converges inside $|z - z_0| < R$. Let's call this power series $g(z)$. Then g is holomorphic and $g(z_0) = a_0$., while $g(z) = f(z)$, $\forall z \neq z_0$.
So we can define $f(z_0) = g(z_0) = a_0$, and by doing this, we have defined a value that the function can take at the singularity at z_0. Therefore, it's a removable singularity. $\square$

This leads to the following

Corollary 12
The function f has an isolated singularity at z_0, which is a pole if and only if

$$\lim_{z \to z_0} |f(z)| = \infty$$

Proof. Part 1 $\Longrightarrow$

$$z_0 \text{ is a pole} \implies \lim_{z \to z_0} |f(z)| = \infty$$

Assume that z_0 is a pole of f. Then for some small punctured disk $D'(z_0, \delta)$, we can write

$$f(z) = (z - z_0)^{-m} h(z)$$

where m is the order of the pole. Here h is holomorphic on the whole disk and it is also non-vanishing.

$$|f(z)| = \left| \frac{1}{(z - z_0)^m} h(z) \right| = \frac{1}{|z - z_0|^m} |h(z)| \to \infty \text{ as } z \to z_0$$

since

$$\lim_{z \to z_0} \frac{1}{|z - z_0|^m} = +\infty$$

and

$$\lim_{z \to z_0} |h(z)| = |h(z_0)| \neq 0 \text{ as } h(z) \neq 0 \text{ on } D$$

Part 2 $\Longleftarrow$

$$\lim_{z \to z_0} |f(z)| = \infty \implies z_0 \text{ is a pole}$$

Looking at $g = \frac{1}{f} : D'(z_0, \delta) \to \mathbb{C}$.

$$|g(z)| = \left| \frac{1}{f(z)} \right| \to 0 \text{ as } z \to z_0$$

Now, since z_0 is an isolated singularity of $g(z)$ and $g(z)$ is bounded on the disk, Riemann's Theorem says that z_0 is a removable singularity. So

$$\lim_{z \to z_0} |g(z)| = 0 \implies \text{ we must define } g(z_0) = 0$$

Then z_0 is a zero of $\frac{1}{f}$, and so, by the definition of a pole, f has a pole at z_0. $\square$

This leads to the following

Classification of Isolated Singularities

- removable singularities
- poles
- essential singularities (everything else)

Example 74
The function $f(z) = e^{\frac{1}{z}}$ has an essential singularity at 0.
Suppose that z is real and that we are approaching 0 from the right (i.e. from the positive real axis, going to the left). Then

$$\lim_{z \to 0^+} \frac{1}{z} = \infty \implies \lim_{z \to 0^+} e^{\frac{1}{z}} = \infty$$

Now suppose that z is still real, but we take the limit from the other side of the real axis, i.e. we are approaching 0 from the left. We have that

$$\lim_{z \to 0^-} \frac{1}{z} = -\infty \implies \lim_{z \to 0^-} e^{\frac{1}{z}} = 0$$

We can keep going. For example, we could take z to be purely imaginary, i.e. $z = ix$, and take x to be positive at approach 0 from the positive imaginary axis, i.e. from above. We get that

$$|e^{\frac{1}{z}}| = |e^{\frac{1}{ix}}| = |e^{\frac{-i}{x}}| = 1 = const$$

so the limit is also 1, which is a constant. Thus we see that we get different limits as we approach the singularity from different directions. This is weird!

6.22 (T) Casorati-Weierstrass

Theorem 37 (Casorati-Weierstrass)
Suppose z_0 is an essential singularity of $f(z)$. Let r be small enough for f to be holomorphic on $D(z_0, r)$. Given $w \in \mathbb{C}$ and $\varepsilon > 0$, $\exists z \in D'(z_0, r)$ s.t. $|f(z) - w| < \varepsilon$.
Then we say that $f(D'(z_0, r))$ is dense in $\mathbb{C}$

Proof. (by contradiction)
We are going to prove the negated statement, by contradiction.

$$\neg[\forall \varepsilon > 0, \exists z \in D'(z_0, r) \text{ s.t. } |f(z) - w| < \varepsilon] = \exists \varepsilon_0 \text{ s.t. } \forall z \in D'(z_0, r), |f(z) - w| \geq \varepsilon_0$$

Consider the function $g : D'(z_0, r) \to \mathbb{C}$ s.t. $g(z) = \frac{1}{f(z) - w}$.
The denominator is never 0 since f is holomorphic on $D'(z_0, r)$ and $f(z) \neq w$ because $|f(z) - w| \geq \varepsilon_0$, we can deduce that g is holomorphic on D' (the denominator is not 0).
Look at $|g(z)| = \frac{1}{|f(z) - w|} \leq \frac{1}{\varepsilon_0}$, $\forall z \in D'$.
Therefore, since g is bounded on the punctured disk, by Riemann's Theorem, g has a removable singularity at z_0 (we can define $g(z_0)$ and g is continuous and holmorphic at

z_0).

$$g(z) = \frac{1}{f(z) - w} \iff f(z) - w = \frac{1}{g(z)} \iff f(z) = w + \frac{1}{g(z)}$$

Case 1: $g(z_0) = 0$

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \left| w + \frac{1}{g(z)} \right| = \lim_{z \to z_0} (w) + \lim_{z \to z_0} \frac{1}{g(z)} = \lim_{z \to z_0} (w) + \frac{1}{g(z_0)} = \infty$$

since $\lim\limits_{z \to z_0} (w)$ is a fixed number, while $\frac{1}{g(z_0)} = \infty$.
Therefore, z_0 is a pole of $f(z)$.

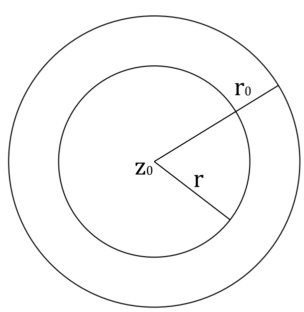
Case 2: $g(z_0) \neq 0$.
g is holomorphic on the whole disk.
We define $f(z_0) = w + \frac{1}{g(z_0)}$. The f becomes holomorphic at z_0, i.e. z_0 is removable.
So in one case, we get an isolated singularity, while in the other case we get a pole. This is a contradiction. □

Note: We can't plot functions close to an essential singularity. It goes everywhere.

6.23 The Mean Value Property and the Maximum Modulus Principle

Discussion
Let f be holomorphic and non-constant on $D(z_0, r)$ and $r < r_0$. Let C be the circle $|z - z_0| < r$ and assume Cauchy's Integral Formula

$$f(z_0) = \frac{1}{2\pi i} \int\limits_C \frac{f(z)}{z - z_0} dz$$



We are observing the integral only on the points on the circle. Summing up the values on the circle is the same as integrating along the circle, and since we are diving by 2π (which is the length of the circle), we get an average value of the function on the circle. Moreover, the average turns out to be the same as the value of the function at z_0. This

is always the case.
If we know the value of $f(z_0)$, we know the average of the function and vice versa.

This is called **The Mean Value Property**
Now consider the following discussion which will form part of the proof for the Maximum Modulus Principle.
Parametrise with $z(t) = z_0 + re^{it}$ where $0 \le t \le 2\pi$. We get that

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ir^{it} dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt$$

$$\begin{aligned} |f(z_0)| &= \left| \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt \right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \left| f(z_0 + re^{it}) \right| dt \end{aligned}$$

This is true for any radius $r \in [0, r_0]$.
Assume that $|f(z)| \le |f(z_0)|$, $\forall z$ with $|z - z_0| = r$. Then we have that

$$\begin{aligned} |f(z_0)| &\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|dt \\ &\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|dz = |f(z_0)| \\ &\implies |f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|dt \\ &\implies 0 = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|dt - |f(z_0)| \\ &\implies 0 = \int_0^{2\pi} |f(z_0 + re^{it})|dt - \int_0^{2\pi} |f(z_0)|dt \\ &\implies 0 = \int_0^{2\pi} \left(|f(z_0 + re^{it})| - |f(z_0)|\right) dt \end{aligned}$$

So we have an integral whose value is 0 and we are also assuming that $|f(z)| \le |f(z_0)| \implies |f(z_0 + re^{it})| - |f(z_0)| \le 0$. (it is also continuous). So this actually implies that $\forall t \in [0, 2\pi]$, we have that

$$|f(z)| = |f(z_0)|, \forall z \text{ with } |z - z_0| = r, \forall r \in [0, r_0]$$

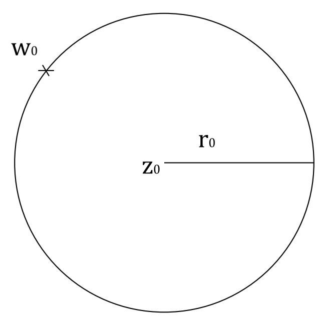
i.e.

$$|f(z)| = |f(z_0)|, \forall z \in D(z_0, r_0)$$

But this is a contradiction since we are assuming that the function is not constant. The contradiction comes from the assumption that $|f(z)| \leq |f(z_0)|$. So we conclude that if f is holomorphic and non-constant in the neighbourhood $D(z_0, r_0)$ at z_0, then there exists at least one point z in it such that $|f(z)| > |f(z_0)|$

Theorem 38 (The Maximum Modulus Principle)
If f is holomorphic and non-constant in a region Ω, then $|f(z)|$ has no maximum value inside Ω.

Proof. We prove the theorem by contradiction.
If we assume that $\exists z_0 \in \Omega$ s.t. $|f(z)| \leq |f(z_0)|$, $\forall z \in D(z_0, r) \subseteq \Omega$, then we get that f is constant on $D(z_0, r)$ by the previous discussion. Then by The Principle of Analytic Continuation, we can conclude that f is constant on the whole of Ω. This contradicts the assumption of the theorem. Thus, we reject this possibility.
Assume that $\Omega = D(z_0, r_0)$. Assume that f is holomorphic on $D(z_0, r)$ and continuous on $\overline{D}(z_0, r_0) = \{z : |z - z_0| \leq R_0\}$. Since $|f(z)|$ is continuous on $\overline{D}(z_0, r_0)$, it is going to achieve a maximum there.
Set $M = \max_{|z-z_0|\leq r_0} |f(z)|$, i.e. $\exists w_0 \in \overline{D}(z_0, r_0)$ s.t. $M = |f(w_0)|$.
By the max modulus principle, $w_0 \notin D(z_0, r_0)$ but $|w - z_0| = r_0$, i.e. w_0 is *on* the circle.



$\square$